R-diagonal pairs - a common approach to Haar unitaries and circular elements

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Introduction

The Haar unitary and the circular element provide two of the most frequently used $*$-distributions in the free probability theory of Voiculescu, and in its remarkable applications to the study of free products of von Neumann algebras (see e.g. [4,5,11,12,19,20]). The starting point of the present paper is that if the R-transform (i.e. the free analogue of the logarithm of the Fourier transform) of these $*$-distributions is considered, then expressions of a similar nature are obtained. The formulas are:

$$(I) \quad [R(\mu_{u,u^*})](z_1,z_2) \;=\; \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(2k-2)!}{(k-1)!k!}(z_1z_2)^k + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(2k-2)!}{(k-1)!k!}(z_2z_1)^k$$

and

$$(II) \qquad [R(\mu_{c,c^*})](z_1,z_2) \;=\; z_1z_2 + z_2z_1,$$

where u is a Haar unitary and c is a circular element (in some non-commutative probability spaces). Hence, a class of pairs of elements which contains both (u,u^*) and (c,c^*) is the following: for x,y elements (random variables) in a non-commutative probability space, the

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pair (x, y) is said to be R-*diagonal* if the R-transform $R(\mu_{x,y})$ has the form:

$$(III) \qquad [R(\mu_{x,y})](z_1, z_2) \ = \ \sum_{k=1}^{\infty} \alpha_k (z_1 z_2)^k + \sum_{k=1}^{\infty} \alpha_k (z_2 z_1)^k$$

for some sequence $(\alpha_k)_{k=1}^{\infty}$ of complex coefficients.

The main result of the paper (stated in Theorem 1.5, Corollary 1.8 below) is that the class of R-diagonal pairs has a remarkable property of "absorption" under the operation of "nested" multiplication of free pairs. That is, for a_1, a_2, p_1, p_2 in a non-commutative probability space: if the sets $\{a_1, a_2\}$ and $\{p_1, p_2\}$ are free, and if the pair (a_1, a_2) is R-diagonal, then so is $(a_1 p_1, p_2 a_2)$; and moreover, there exists a simple formula relating the α's of Eqn.(III) written for the two pairs (a_1, a_2) and $(a_1 p_1, p_2 a_2)$.

As an immediate consequence, R-diagonal pairs exist in abundance, even if we only want to consider pairs of the form (x, x^*), and in the W*-probabilistic context. We hope that, while not as fundamental as (u, u^*) and (c, c^*) from Eqns.(I), (II), other pairs in this class will also find their role in the theory, and in its applications.

A special interest is presented by the R-diagonal pairs of the form $(up, (up)^*)$, where u is a Haar unitary and p is $*$-free from u (in a C^*-probability space, say). A couple of applications of the main result to this class is presented in Sections 1.9, 1.10 below. A situation when several such free pairs $(up_1, (up_1)^*), \ldots, (up_k, (up_k)^*)$ are considered at the same time is addressed in Theorem 1.13.

¿From the technical point of view, our approach to the R-diagonal pairs is based on the combinatorial description of the R-transform, via the lattice $NC(n)$ of non-crossing partitions of $\{1, \ldots, n\}$, $n \geq 1$. More than once the proofs depend in an essential way on considerations involving a certain operation $\boxed{\star}$ on formal power series, introduced in our previous paper [10]; this operation represents in some sense the combinatorial facet of the R-transform approach to the multiplication of free n-tuples of non-commutative random variables.

A detailed description of the results of the paper is made in the next-coming Section 1. The rest of the paper is organized as follows. In Section 2 we review some facts about non-crossing partitions, and in Section 3 we review the R-transform and the operation $\boxed{\star}$. Section 4 is devoted to pointing out a certain canonical bijection between the set of intervals of $NC(n)$ and the set of 2-divisible partitions in $NC(2n)$, which plays an important role in our considerations; we also note in Section 4 how, as a consequence of this combinatorial fact, one of the applications of $\boxed{\star}$ presented in [10] can be improved. The proofs of the results on R-diagonal pairs announced in Section 1 are divided between the remaining Sections 5-8

of the paper.

1. Presentation of the results

1.1 Basic definitions In this section we briefly review some basic free probabilistic terminology used throughout the paper (for a more detailed treatment, we refer to the monograph [21]).

The framework We will call *non-commutative probability space* a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital algebra (over $\mathbf{C}$), and $\varphi : \mathcal{A} \to \mathbf{C}$ is a linear functional normalized by $\varphi(1) = 1$. If we require in addition that $\mathcal{A}$ is a C^*-algebra, and φ is positive, then $(\mathcal{A}, \varphi)$ is called a *C^*-probability space.*

Freeness A family of unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_n \subseteq \mathcal{A}$ is said to be free in $(\mathcal{A}, \varphi)$ if for every $k \geq 1$, $1 \leq i_1, \ldots, i_k \leq n$, $a_1 \in \mathcal{A}_{i_1}, \ldots, a_k \in \mathcal{A}_{i_k}$, we have the implication:

$$\left\{ \begin{array}{c} i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{k-1} \neq i_k \\ \varphi(a_1) = \varphi(a_2) = \cdots = \varphi(a_k) = 0 \end{array} \right\} \Rightarrow \varphi(a_1 a_2 \cdots a_k) = 0. \tag{1.1}$$

The notion of freeness in $(\mathcal{A}, \varphi)$ extends to arbitrary subsets of $\mathcal{A}$, by putting $\mathcal{X}_1, \ldots, \mathcal{X}_n \subseteq \mathcal{A}$ to be free if and only if the unital subalgebras generated by them are so. The freeness of a family of elements $x_1, \ldots, x_n \in \mathcal{A}$ is defined as the one of the family of subsets $\{x_1\}, \ldots, \{x_n\} \subseteq \mathcal{A}$.

If $(\mathcal{A}, \varphi)$ is a C^*-probability space, then the fact that $x_1, \ldots, x_n \in \mathcal{A}$ are $*$-free means by definition that the subsets $\{x_1, x_1^*\}, \ldots, \{x_n, x_n^*\}$ are free.

Joint distributions The *joint distribution* of the family of elements $a_1, \ldots, a_n \in \mathcal{A}$, in the non-commutative probability space $(\mathcal{A}, \varphi)$, is by definition the linear functional $\mu_{a_1,\ldots,a_n} : \mathbf{C}\langle X_1, \ldots, X_n \rangle \to \mathbf{C}$ given by:

$$\left\{ \begin{array}{lcl} \mu_{a_1,\ldots,a_n}(1) & = 1, & \\ \mu_{a_1,\ldots,a_n}(X_{i_1} \cdots X_{i_k}) & = & \varphi(a_{i_1} \cdots a_{i_k}) \quad \text{for } k \geq 1 \text{ and } 1 \leq i_1, \ldots, i_k \leq n, \end{array} \right. \tag{1.2}$$

where $\mathbf{C}\langle X_1, \ldots, X_n \rangle$ is the algebra of polynomials in n non-commuting indeterminates $X_1, \ldots, X_n$.

If we only have one element $a = a_1 \in \mathcal{A}$, then the functional in (1.2) is just $\mu_a : \mathbf{C}[X] \to \mathbf{C}$, $\mu_a(f) = \varphi(f(a))$ for $f \in \mathbf{C}[X]$, and is called the *distribution* of a in $(\mathcal{A}, \varphi)$.

If $(\mathcal{A}, \varphi)$ is a non-commutative probability space, then when speaking about the $*$-*distribution* of an element $x \in \mathcal{A}$, one usually refers to the joint distribution μ_{x,x^*}; another (equivalent) approach goes by looking at the joint distribution $\mu_{Re(x),\ Im(x)}$ of the real and imaginary parts $Re(x) = (x + x^*)/2$, $Im(x) = (x - x^*)/2i$.

The definitions of a Haar unitary and of a circular element are made by prescribing (for the circular in an indirect way) what is their $*$-distribution. We will consider the framework of a C^*-probability space $(\mathcal{A}, \varphi)$. Recall that:

- an element $u \in \mathcal{A}$ is called a *Haar unitary* in $(\mathcal{A}, \varphi)$ if it is unitary, and if $\varphi(a^n) = 0$ for every $n \in \mathbf{Z} \setminus \{0\}$;

- an element $a \in \mathcal{A}$ is called *semicircular* in $(\mathcal{A}, \varphi)$ if it is selfadjoint and its distribution μ_a is $\frac{1}{2\pi}\sqrt{4-t^2}dt$ on [-2,2] (in other words, if $\varphi(a^n) = \frac{1}{2\pi}\int_{-2}^{2} t^n \sqrt{4-t^2}dt$, $n \geq 0$);

- an element $c \in \mathcal{A}$ is called *circular* in $(\mathcal{A}, \varphi)$ if it is of the form $(a + ib)/\sqrt{2}$ with a, b semicircular and free in $(\mathcal{A}, \varphi)$.

We take this occasion to review one more remarkable distribution:

- an element $a \in \mathcal{A}$ is called *quarter-circular* in $(\mathcal{A}, \varphi)$ if it is positive and its distribution μ_a is $\frac{1}{\pi}\sqrt{4-t^2}dt$ on [0,2] (i.e., if $\varphi(a^n) = \frac{1}{\pi}\int_0^2 t^n\sqrt{4-t^2}dt$, $n \geq 0$).

The R-transform Given a functional of the kind appearing in Eqn.(1.2) (i.e. $\mu : \mathbf{C}\langle X_1, \ldots, X_n\rangle \to \mathbf{C}$, such that $\mu(1) = 1$), its R-transform $R(\mu)$ is a certain formal power series in n "non-commuting complex variables" $z_1, \ldots, z_n$:

$$[R(\mu)](z_1, \ldots, z_n) \;=\; \sum_{k=1}^{\infty} \sum_{i_1,\ldots,i_k=1}^{n} \alpha_{(i_1,\ldots,i_k)} z_{i_1} \cdots z_{i_k}. \tag{1.3}$$

The coefficients $(\alpha_{(i_1,\ldots,i_k)})_{k\geq 1, 1\leq i_1,\ldots,i_k\leq n}$ of $R(\mu)$ are also called the *free* (or *non-crossing) cumulants* of μ. The precise definition of $R(\mu)$ (i.e. of how the free cumulants are constructed from μ) will be reviewed in Section 3 below.

If $a_1, \ldots, a_n$ are elements in the non-commutative probability space $(\mathcal{A}, \varphi)$, then the R-transform $R(\mu_{a_1,\ldots,a_n})$ contains the information about the joint distribution $\mu_{a_1,\ldots,a_n}$ re-arranged in such a way that the freeness or non-freeness of $a_1, \ldots, a_n$ becomes transparent. More precisely, as proved in [14,8] $a_1, \ldots, a_n$ are free in $(\mathcal{A}, \varphi)$ if and only if the coefficient of $z_{i_1} z_{i_2} \cdots z_{i_k}$ in $[R(\mu_{a_1,\ldots,a_n})](z_1, \ldots, z_n)$ vanishes whenever we don't have $i_1 = i_2 = \cdots = i_k$; i.e., if and only if $R(\mu_{a_1,\ldots,a_n})$ is of the form

$$[R(\mu_{a_1,\ldots,a_n})](z_1, \ldots, z_n) \;=\; f_1(z_1) + \cdots + f_n(z_n), \tag{1.4}$$

for some formal power series of one variable $f_1, \ldots, f_n$.

We will refer to the coefficient-vanishing condition presented in the preceding paragraph by saying that "the series $R(\mu_{a_1,\ldots,a_n})$ has no mixed coefficients". If this happens, then $f_1, \ldots, f_n$ of (1.4) can only be the 1-dimensional R-transforms $R(\mu_{a_1}), \ldots, R(\mu_{a_n})$, respectively.

1.2 Haar unitaries and circular elements If F_k denotes the free group on generators $g_1, \ldots, g_k$, then the left-translation operators $u_1, \ldots, u_k$ with $g_1, \ldots, g_k$ on $l^2(F_k)$ form a family of $*$-free Haar unitaries in $(L(F_k), \tau)$, where $L(F_k)$ is the von Neumann II_1 factor of F_k and τ is the unique normalized trace on $L(F_k)$; $u_1, \ldots, u_k$ is in some sense "the obvious system of generators" for $L(F_k)$.

In recent work of Voiculescu, Radulescu, Dykema (see e.g. [4,5,11,12,20]) it was shown that very powerful results on $L(F_k)$ can be obtained by using a different family of generators, consisting of free semicircular elements. The Haar unitaries are also appearing in this picture, but in a more subtle way, either via asymptotic models (for instance in [19], Section 3), or via the theorem of Voiculescu [20] on the polar decomposition of the circular element. This theorem states that if $\mathcal{A}$ is a von Neumann algebra, with $\varphi : \mathcal{A} \to \mathbf{C}$ a faithful normal trace, and if c is circular in $(\mathcal{A}, \varphi)$, then by taking the polar decomposition $c = up$ of c one gets that: u is a Haar unitary, p is quarter-circular, and u, p are $*$-free. The original proof given by Voiculescu in [20] for this fact depends on the asymptotic matrix model for free semicircular families developed in [19]. A direct, combinatorial proof was recently found by Banica [1].

The starting point of this work was the observation that the R-transforms of the $*$-distributions of the Haar unitary and of the circular element have similar forms. In fact, the goal of the present paper is in some sense to understand the relation between these two elements, from the point of view of the R-transform. Although this will not be our main concern, a new proof for the polar decomposition of the circular element will also follow (see the discussion in 1.9, 1.10 below.)

If c is a circular element in the C^*-probability space $(\mathcal{A}, \varphi)$, then from the fact that its real and imaginary parts are multiples of free semicirculars, one gets immediately that:

$$[R(\mu_{Re(c),\ Im(c)})](z_1, z_2) \ = \ \frac{1}{2}(z_1^2 + z_2^2); \tag{1.5}$$

then by using a result from [8] concerning linear changes of coordinates, this implies:

$$[R(\mu_{c,c^*})](z_1, z_2) \ = \ z_1 z_2 + z_2 z_1. \tag{1.6}$$

On the other hand, it was shown in [15, Section 3.4] that for u a Haar unitary in some C^*-probability space, one has:

$$[R(\mu_{u,u^*})](z_1,z_2) \ = \ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(2k-2)!}{(k-1)!k!}(z_1z_2)^k + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(2k-2)!}{(k-1)!k!}(z_2z_1)^k. \quad (1.7)$$

Thus a class of pairs of elements which contains both (c,c^*) and (u,u^*) is given by the following

1.3 Definition: Let $(\mathcal{A},\varphi)$ be a non-commutative probability space, and let a_1,a_2 be in $\mathcal{A}$. We will say that (a_1,a_2) is an *R-diagonal pair* if

(i) the coefficients of $\underbrace{z_1z_2\cdots z_1z_2}_{2n}$ and $\underbrace{z_2z_1\cdots z_2z_1}_{2n}$ in $[R(\mu_{a_1,a_2})](z_1,z_2)$ are equal, for every $n\geq 1$;

(ii) every coefficient of $R(\mu_{a_1,a_2})$ not of the form mentioned in (i) is equal to 0.

In other words, the pair (a_1,a_2) is R-diagonal if and only if $R(\mu_{a_1,a_2})$ has the form

$$[R(\mu_{a_1,a_2})](z_1,z_2) \ = \ \sum_{k=1}^{\infty} \alpha_k(z_1z_2)^k + \sum_{k=1}^{\infty} \alpha_k(z_2z_1)^k \quad (1.8)$$

for some sequence $(\alpha_k)_{k=1}^{\infty}$. If this happens, then the series of one variable $f(z)=\sum_{k=1}^{\infty}\alpha_k z^k$ will be called the *determining series* of the pair (a_1,a_2).

1.4 Remark From (1.6) it follows that the determining series of (c,c^*) is just $f(z)=z$. The determining series for (u,u^*), coming out from (1.7), will be denoted by $Moeb$:

$$Moeb(z) \ = \ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(2k-2)!}{(k-1)!k!} z^k, \quad (1.9)$$

and will be called the Moebius series (of one variable - compare also to Eqn.(3.8) below).

The main result of the paper can then be stated as follows.

1.5 Theorem Let $(\mathcal{A},\varphi)$ be a non-commutative probability space, such that φ is a trace (i.e. $\varphi(xy)=\varphi(yx)$, $x,y\in\mathcal{A}$), and let $a_1,a_2,p_1,p_2\in\mathcal{A}$ be such that (a_1,a_2) is an R-diagonal pair, and such that $\{p_1,p_2\}$ is free from $\{a_1,a_2\}$. Then (a_1p_1,p_2a_2) is also an R-diagonal pair.

Moreover, there exists a simple formula which connects the determining series of (a_1, a_2) and (a_1p_1, p_2a_2); this formula is presented in Corollary 1.8 below.

1.6 The operation $\boxed{\star}$ The formula announced in the previous phrase involves a certain binary operation $\boxed{\star}$ on the set of formal power series $\{f \mid f(z) = \sum_{k=1}^{\infty} \alpha_k z^k;\ \alpha_1, \alpha_2, \alpha_3, \ldots \in \mathbf{C}\}$. One possible way of defining $\boxed{\star}$ is via the equation

$$R(\mu_{ab}) \ = \ R(\mu_a) \boxed{\star} R(\mu_b), \tag{1.10}$$

holding whenever a is free from b in some non-commutative probability space $(\mathcal{A}, \varphi)$. (This definition makes sense because $R(\mu_{ab})$ is completely determined by $R(\mu_a)$ and $R(\mu_b)$, and because any two series f, g of the considered type [1] can be realized as $R(\mu_a)$ and $R(\mu_b)$ with a, b free in some $(\mathcal{A}, \varphi)$.) The operation $\boxed{\star}$ also has an alternative combinatorial definition, which will be reviewed in Section 3.3 below. The best point of view seems to be to consider both approaches to $\boxed{\star}$, and switch from one to the other as needed. This operation appeared (under a different name) in [14,9], in connection to the work of Voiculescu [18] on products of free elements. The name $\boxed{\star}$ was first used in [10], where the multivariable versions of the operation were introduced and applied. We mention that an important feature distinguishing the 1-dimensional instance of $\boxed{\star}$ from the others is that in this (and only this) case $\boxed{\star}$ is commutative.

The formula announced immediately after Theorem 1.5 comes out in the following way.

1.7 Proposition Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, such that φ is a trace, and let (a_1, a_2) be an R-diagonal pair in $(\mathcal{A}, \varphi)$. If f is the determining series of (a_1, a_2), then: $f \ = \ R(\mu_{a_1a_2}) \boxed{\star}\, Moeb$, where $Moeb$ is the Moebius series, as in Eqn.(1.9) (and of course, $\mu_{a_1a_2} : \mathbf{C}[X] \to \mathbf{C}$ denotes the distribution of the product $a_1 \cdot a_2$).

1.8 Corollary In the context of Theorem 1.5, if f and g are the determining series of the R-diagonal pairs (a_1, a_2) and (a_1p_1, p_2a_2), respectively, then we have the relation

$$g \ = \ f \boxed{\star} R(\mu_{p_1p_2}). \tag{1.11}$$

[1] In this paper the R-transform of an 1-dimensional distribution is viewed as the particular case $n = 1$ of the Eqn.(1.3); we warn the reader that this differs by a factor of z from the notation used in [21].

Proof We can write that

$$g = R(\mu_{a_1p_1p_2a_2}) \boxed{\star}\, Moeb \text{ (by Proposition 1.7)}$$

$$= R(\mu_{a_2a_1p_1p_2}) \boxed{\star}\, Moeb \text{ (because } \varphi \text{ is a trace)}$$

$$= R(\mu_{a_2a_1}) \boxed{\star}\, R(\mu_{p_1p_2}) \boxed{\star}\, Moeb \text{ (by (1.10))}$$

$$= (R(\mu_{a_2a_1}) \boxed{\star}\, Moeb) \boxed{\star}\, R(\mu_{p_1p_2}) \text{ (because } \boxed{\star} \text{ is commutative)}$$

$$= f \boxed{\star}\, R(\mu_{p_1p_2}) \text{ (again by Proposition 1.7).}$$ **QED**

1.9 Application Note that the Moebius series appears in two different ways in the above discussion (in Remark 1.4 and Proposition 1.7, respectively). This has a consequence concerning R-diagonal pairs of the form (x, x^*), in the C^*-context. Let us denote by $\mathcal{R}_c$ the set of formal power series of one variable which occur as $R(\mu)$, with μ a probability measure on $\mathbf{R}$ having compact support contained in $[0, \infty)$ (in connection to $\mathcal{R}_c$, see also [17], Section 3). We have the following

Fact: A formal power series f of one variable can appear as determining series for an R-diagonal pair (x, x^*) in some C^*-probability space if and only if it is of the form $f = g\boxed{\star}Moeb$, with $g \in \mathcal{R}_c$. If this happens, then f can be in fact written as the determining series of an R-diagonal pair $(up, (up)^*)$, with u Haar unitary, p positive, and such that u is $*$-free from p.

Proof Implication "$\Rightarrow$" follows from Proposition 1.7 ($f = R(\mu_{xx^*})\boxed{\star}Moeb$, and $R(\mu_{xx^*}) \in \mathcal{R}_c$). Conversely, assume that $f = g\boxed{\star}Moeb$, with $g \in \mathcal{R}_c$. We can always find a C^*-probability space $(\mathcal{A}, \varphi)$ and $u, p \in \mathcal{A}$, $*$-free, such that u is Haar unitary, p is positive, and $R(\mu_{p^2}) = g$. (For instance we can take $\mathcal{A} = L^\infty(\mu) \star L^\infty(\mathbf{T})$, endowed with the free product of μ with the Lebesgue measure on $\mathbf{T}$, where μ is such that $R(\mu) = g$.) The pair $(up, (up)^*)$ is R-diagonal by Theorem 1.5, and has determining series $Moeb\boxed{\star}R(\mu_{p^2}) = Moeb\boxed{\star}g = f$, by Corollary 1.8. **QED**

We thus see that, from the point of view of the $*$-distribution, any R-diagonal pair (x, x^*) in a C^*-probability space can be replaced with one of the form $(up, (up)^*)$. This can be pushed to a "polar decomposition result", if we consider the von Neumann algebra setting, with a normal faithful trace, and if we also assume that Ker $x = \{0\}$. Indeed, in such a situation we get that the von Neumann subalgebras generated by x and up (in their

W^*-probability spaces) are canonically isomorphic, by an isomorphism which sends x into up (this is the same type of argument as, for instance, in [20], Remark 1.10). We obtain in this way a product decomposition $x = u'p'$, with u' Haar unitary, p' positive, and u' $*$-free from p', and the uniqueness of the polar decomposition shows that $u'p'$ is necessarily the polar decomposition of x. (The needed fact that Ker $p' = \{0\}$ is obtained by verifying that the distribution of p'^2 has no atom at 0.)

1.10 Application Let $(\mathcal{A}, \varphi)$ be a C^*-probability space, with φ a trace, and let $u, p \in \mathcal{A}$ be such that u is a Haar unitary, $*$-free from p. We look for necessary and sufficient conditions for the real and imaginary parts of up to be free. Without loss of generality, we can assume that p is normalized in such a way that $\varphi(pp^*) = 1$.

By using 1.5, 1.8 and also Eqn.(1.7) of 1.2 we infer that (up, p^*u^*) is an R-diagonal pair, with determining series g given by:

$$g \;=\; Moeb \boxed{\star}\, R(\mu_{pp^*}). \tag{1.12}$$

We write explicitly $g(z) = \sum_{k=1}^{\infty} \beta_k z^k$; the assumption that $\varphi(pp^*) = 1$ plugged into (1.12) implies that $\beta_1 = 1$. Remembering how the determining series was defined in 1.3, we have that

$$[R(\mu_{up,p^*u^*})](z_1, z_2) \;=\; \sum_{k=1}^{\infty} \beta_k (z_1 z_2)^k \;+\; \sum_{k=1}^{\infty} \beta_k (z_2 z_1)^k;$$

then by doing a linear change of coordinates (as in [8], Section 5) we get

$$[R(\mu_{Re(up),Im(up)})](z_1, z_2) \;=\; \sum_{k=1}^{\infty} \frac{\beta_k}{4^k} ((z_1 + iz_2)(z_1 - iz_2))^k \;+\; \sum_{k=1}^{\infty} \frac{\beta_k}{4^k} ((z_1 - iz_2)(z_1 + iz_2))^k. \tag{1.13}$$

By the result stated in (1.4), $Re(up)$ and $Im(up)$ are free if and only if the series in (1.13) has no mixed coefficients; but a direct analysis of the right-hand side of (1.13) shows that this can happen if and only if $\beta_2 = \beta_3 = \cdots = 0$. Hence $Re(up)$ and $Im(up)$ are free if and only if the series g of (1.12) is just $g(z) = z$. Finally, the equation $[Moeb \boxed{\star}\, R(\mu_{pp^*})](z) = z$ is easily solved "in the unknown" $R(\mu_{pp^*})$, and is found to be equivalent to $[R(\mu_{pp^*})](z) = z/(1-z)$. We thus obtain the following

Fact: With u and p as in the first paragraph of 1.10, the necessary and sufficient condition for the real and imaginary part of up to be free is

$$[R(\mu_{pp^*})](z) = z/(1-z). \tag{1.14}$$

Note that if $\beta_1 = 1$ and $\beta_2 = \beta_3 = \cdots = 0$, then the right-hand side of (1.13) is just $(z_1^2 + z_2^2)/2$, and by comparing (1.13) against (1.5) we find that up is circular. Hence up is circular whenever (1.14) holds.

We also note that (1.14) does hold if $p = p^* =$ quarter-circular (the square of the quarter-circular is the same thing as the square of the semicircular, and the R-transform of the latter square is well-known - see e.g. [10], Lemma 4.2 or Lemma 1.1 in the Appendix). Here again the uniqueness of the polar decomposition yields from this point a proof for the polar decomposition of the circular element.

1.11 Case of several pairs Another aspect which can be studied in the context of 1.10 is: what happens if instead of looking just at up, we look at a family $up_1, up_2, \ldots, up_k$, where $u, p_1, p_2, \ldots, p_k$ are $*$-free? It is shown by Banica in [1] that $up_1, up_2, \ldots, up_k$ are also $*$-free if the following happens: every p_j $(1 \leq j \leq k)$ is in some sense $*$-modeled by an operator of the form $S^* + S^{m_j}$, $m_j \geq 1$, where S is the unilateral shift on $l^2(\mathbf{N})$. We have found a condition expressed in terms of "moments of pairs", which still implies the $*$-freeness of $up_1, up_2, \ldots, up_k$. This condition, which is satisfied by the pair $(S^* + S^m, S + S^{m*})$ for every $m \geq 1$, is described as follows.

1.12 Definition Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and let a_1, a_2 be in $\mathcal{A}$. We will say that (a_1, a_2) is a *diagonally balanced pair* if

$$\varphi(\underbrace{a_1 a_2 \cdots a_1 a_2 a_1}_{2n+1}) = \varphi(\underbrace{a_2 a_1 \cdots a_2 a_1 a_2}_{2n+1}) = 0, \quad \text{for every } n \geq 0. \tag{1.15}$$

Then we have:

1.13 Theorem Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, with φ a trace, and let $u, p_{1,1}, p_{1,2}, \ldots, p_{k,1}, p_{k,2}$ be in $\mathcal{A}$ such that:

(i) u is invertible and $\varphi(u^n) = 0$ for every $n \in \mathbf{Z} \setminus \{0\}$;

(ii) the pairs $(p_{1,1}, p_{1,2}), \ldots, (p_{k,1}, p_{k,2})$ are diagonally balanced;

(iii) the sets $\{u, u^{-1}\}, \{p_{1,1}, p_{1,2}\}, \ldots, \{p_{k,1}, p_{k,2}\}$ are free.

Then the sets $\{up_{1,1}, p_{1,2}u^{-1}\}, \ldots, \{up_{k,1}, p_{k,2}u^{-1}\}$ are also free.

The techniques used for proving the results announced in 1.5, 1.7, 1.13 above are based on the combinatorial approach to the R-transform, via the lattice $NC(n)$ of non-crossing

partitions of $\{1, \ldots, n\}$, $n \geq 1$. In some instances, we are able to use an elegant idea of Biane [2] which reduces assertions on non-crossing partitions to calculations in the group algebra of the symmetric group (but there are also situations when this mechanism is apparently not applying, and we have to use "geometric" arguments). It seems that from the combinatorial point of view, a certain canonical bijection between the set of intervals of $NC(n)$ and the set of 2-divisible partitions in $NC(2n)$ has a significant role in the considerations. We have incidentally noticed that, as a consequence of this combinatorial fact, we can improve one of the applications of the operation $\boxed{\star}$ that were presented in [10]. Namely, we have:

1.14 Theorem Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, with φ a trace, and let $a_1, \ldots, a_n, b', b'' \in \mathcal{A}$ be such that the pair (b', b'') is diagonally balanced, and such that $\{b', b''\}$ is free from $\{a_1, \ldots, a_n\}$. Then $\{b'a_1b'', \ldots, b'a_nb''\}$ is free from $\{a_1, \ldots, a_n\}$.

The particular case of 1.14 when $b' = b''$ is a semicircular element is stated in Application 1.10 of [10]. The Theorem 1.14 sounds quite "elementary", and it is not impossible that it also has a simple direct proof, using only the definition of freeness (we weren't able to find one, though).

2. Preliminaries on non-crossing partitions

2.1 Definition of NC(n) If $\pi = \{B_1, \ldots, B_r\}$ is a partition of $\{1, \ldots, n\}$ (i.e. $B_1, \ldots, B_r$ are pairwisely disjoint, non-void sets, such that $B_1 \cup \cdots \cup B_r = \{1, \ldots, n\}$), then the equivalence relation on $\{1, \ldots, n\}$ with equivalence classes $B_1, \ldots, B_r$ will be denoted by $\stackrel{\pi}{\sim}$; the sets $B_1, \ldots, B_r$ will be also referred to as the *blocks* of π. The number of elements in the block B_k, $1 \leq k \leq r$, will be denoted by $|B_k|$.

A partition π of $\{1, \ldots, n\}$ is called *non-crossing* if for every $1 \leq i < j < i' < j' \leq n$ such that $i \stackrel{\pi}{\sim} i'$ and $j \stackrel{\pi}{\sim} j'$, it necessarily follows that $i \stackrel{\pi}{\sim} j \stackrel{\pi}{\sim} i' \stackrel{\pi}{\sim} j'$. The set of all non-crossing partitions of $\{1, \ldots, n\}$ will be denoted by $NC(n)$. On $NC(n)$ we will consider the *refinement order,* defined by $\pi \leq \rho \stackrel{def}{\Leftrightarrow}$ each block of ρ is a union of blocks of π (in other words, $\pi \leq \rho$ means that the implication $i \stackrel{\pi}{\sim} j \Rightarrow i \stackrel{\rho}{\sim} j$ holds, for $1 \leq i, j \leq n$). The partially ordered set $NC(n)$ was introduced by G. Kreweras in [7], and its combinatorics has been studied by several authors (see e.g. [13], and the list of references there).

2.2 The circular picture of a partition $\pi = \{B_1, \ldots, B_r\}$ of $\{1, \ldots, n\}$ is obtained by drawing n equidistant and clockwisely ordered points $P_1, \ldots, P_n$ on a circle, and then by drawing for each block B_k of π the inscribed convex polygon with vertices $\{P_i \mid i \in B_k\}$ (this polygon may of course be reduced to a point or a line segment). It is immediately verified that π is non-crossing if and only if the r convex polygons obtained in this way are disjoint.

We take the occasion to mention that when B is a block of the partition $\pi \in NC(n)$, we will use for $i < j$ in B the expression "i and j are consecutive in B" to mean that either $B \cap \{i+1, \ldots, j-1\} = \emptyset$ or $i = \min B$, $j = \max B$. On the circular picture, the fact that i, j are consecutive in B means that P_iP_j is an edge (rather than a diagonal) of the inscribed polygon with vertices $\{P_h \mid h \in B\}$.

2.3 The Kreweras complementation map is a remarkable order anti-isomorphism $K : NC(n) \to NC(n)$, introduced in [7], Section 3, and described as follows.

Let π be in $NC(n)$, and consider the circular picture of π, involving the points $P_1, \ldots, P_n$, as in 2.2. Denote the midpoints of the arcs of circle $P_1P_2, \ldots, P_{n-1}P_n,\ P_nP_1$ by $Q_1, \ldots, Q_{n-1}$, Q_n, respectively. Then the "complementary" partition $K(\pi) \in NC(n)$ is given, in terms of the corresponding equivalence relation $\overset{K(\pi)}{\sim}$ on $\{1, \ldots, n\}$, by putting for $1 \le i, j \le n$:

$$i \overset{K(\pi)}{\sim} j \ \overset{def}{\Leftrightarrow} \ \left\{ \begin{array}{l} \text{there are no } 1 \le h, k \le n \text{ such that } h \overset{\pi}{\sim} k \text{ and such that} \\ \text{the line segments } Q_iQ_j \text{ and } P_hP_k \text{ have non-void intersection.} \end{array} \right. \tag{2.1}$$

It is easily verified that $\overset{K(\pi)}{\sim}$ of (2.1) is indeed an equivalence relation on $\{1, \ldots, n\}$, and that the partition $K(\pi)$ corresponding to it is non-crossing (for the latter thing, we just have to look at the circular picture of $K(\pi)$, with respect to the points $Q_1, \ldots, Q_n$).

In order to verify that $K : NC(n) \to NC(n)$ defined in this way really is an order anti-isomorphism, one can first remark that $K^2(\pi)$ is (for every $\pi \in NC(n)$) a rotation of π with $360^o/n$; this shows in particular that K is a bijection. The implication $\pi \le \rho \Rightarrow K(\pi) \ge K(\rho)$ is a direct consequence of (2.1), and the converse must also hold, since K^2 is an order-preserving isomorphism of $NC(n)$.

As a concrete example, the circular-picture verification for $K(\{\{1, 4, 5\}, \{2, 3\},\ \{6, 8\}, \{7\}\}) = \{\{1, 3\}, \{2\}, \{4\}, \{5, 8\}, \{6, 7\}\} \in NC(8)$ is shown in Figure 1.

Figure 1.

By examining the common circular picture for π and $K(\pi)$, it becomes quite obvious that $K(\pi)$ could be alternatively defined as the biggest ρ in $(NC(n), \leq)$ with the property that the partition of $\{1, 2, \ldots, 2n\}$ obtained by interlacing π and ρ is still non-crossing. This fact is formally recorded in the next proposition.

2.4 Proposition Let π and ρ be in $NC(n)$. Denote by π' and ρ' the partitions of $\{2, 4, \ldots, 2n\}$ and $\{1, 3, \ldots, 2n-1\}$, respectively, which get identified to π and ρ via the order-preserving bijections $\{1, \ldots, n\} \to \{2, 4, \ldots, 2n\}$ and $\{1, \ldots, n\} \to \{1, 3, \ldots, 2n-1\}$. Denote by σ the partition of $\{1, 2, \ldots, 2n\}$ formed by putting π' and ρ' together. Then σ is non-crossing if and only if $\pi \leq K(\rho)$.

2.5 The relative Kreweras complement Given $\rho \in NC(n)$, one can define a relativized version of the Kreweras complementation map, K_ρ, which is an order anti-isomorphism of $\{\pi \in NC(n) \mid \pi \leq \rho\}$. If we write explicitly $\rho = \{B_1, \ldots, B_r\}$, then an arbitrary element of $\{\pi \in NC(n) \mid \pi \leq \rho\}$ can be written as $\{A_{1,1}, \ldots, A_{1,s_1}, \ldots, A_{r,1}, \ldots, A_{r,s_r}\}$, where $A_{k,1} \cup \cdots \cup A_{k,s_k} = B_k$, $1 \leq k \leq r$. The relative Kreweras complement $K_\rho(\pi)$ is obtained by looking for every $1 \leq k \leq r$ at the non-crossing partition $\{A_{k,1}, \ldots, A_{k,s_k}\}$ of B_k, and by taking its Kreweras complement, call it θ_k, in the sense of Section 2.3 above (of course, in order to do this, we need to identify canonically B_k with $\{1, \ldots, |B_k|\}$). Then $K_\rho(\pi)$ is obtained by putting together the partitions θ_1 of B_1, $\ldots$, θ_r of B_r (see also [10], Sections 2.4, 2.5).

Note that if $\rho = \{\{1, \ldots, n\}\}$ is the maximal element of $(NC(n), \leq)$, then K_ρ coincides with $K : NC(n) \to NC(n)$ discussed in 2.3.

2.6 Relation with permutations A useful way of "encoding" non-crossing partitions by permutations was introduced by Ph. Biane in [2]. Let $\mathcal{S}_n$ denote the group of all permutations of $\{1, \ldots, n\}$. For $B = \{i_1 < i_2 < \cdots < i_m\} \subseteq \{1, \ldots, n\}$ we denote by $\gamma_B \in \mathcal{S}_n$ the cycle given by

$$\begin{cases} \gamma_B(i_1) = i_2, \ldots, \gamma_B(i_{m-1}) = i_m, \gamma_B(i_m) = i_1, \\ \gamma_B(j) = j \text{ for } j \in \{1, \ldots, n\} \setminus B \end{cases} \tag{2.2}$$

(if $|B| = 1$, we take γ_B to be the unit of $\mathcal{S}_n$). Then for every $\pi \in NC(n)$, the permutation associated to it is

$$Perm(\pi) \ = \prod_{B \ block \ of \ \pi} \gamma_B \tag{2.3}$$

(the cycles (γ_B ; B block of π) commute, so the order of the factors in the product (2.3) does not matter).

It was shown in [2] how the (obviously injective) map $Perm : NC(n) \to \mathcal{S}_n$ can be used for an elegant analysis of the skew-automorphisms (i.e. automorphisms or anti-automorphisms) of $(NC(n), \leq)$. We will only need here the "$Perm$" characterization of the Kreweras complementation map, which goes as follows: if $\gamma \in \mathcal{S}_n$ denotes the cycle $(1 \to 2 \to \cdots \to n \to 1)$, then

$$Perm(K(\pi)) \ = \ Perm(\pi)^{-1} \ \gamma, \text{ for every } \pi \in NC(n). \tag{2.4}$$

This characterization can be extended without difficulty to the situation of the relative Kreweras complement, we have:

$$Perm(K_\rho(\pi)) \ = \ Perm(\pi)^{-1} \ Perm(\rho), \tag{2.5}$$

for every $\pi, \rho \in NC(n)$ such that $\pi \leq \rho$ (see [10], Section 2.5).

3. Review of the operation $\boxed{\star}$ and of the R-transform

We will follow the presentation of [10], Section 3. The approach to the R-transform taken here is based on elements of Moebius inversion theory for non-crossing partitions, on the lines of [14]. We mention that the connection between this approach and the original one

of Voiculescu in [17] is made via an alternative description of the n-dimensional R-transform, which goes by "modeling on the full Fock space over $\mathbf{C}^n$" (see [8]).

3.1 Notation Let n be a positive integer. We denote by Θ_n the set of formal power series without constant coefficient in n non-commuting variables $z_1, \ldots, z_n$. An element of Θ_n is thus a series of the form

$$f(z_1, \ldots, z_n) \ = \ \sum_{k=1}^{\infty} \ \sum_{i_1, \ldots, i_k = 1}^{n} \alpha_{(i_1, \ldots, i_k)} z_{i_1} \cdots z_{i_k}, \tag{3.1}$$

where $(\alpha_{(i_1, \ldots, i_k)} \ ; \ k \geq 1, \ 1 \leq i_1, \ldots, i_k \leq n)$ is a family of complex coefficients.

3.2 Notations for coefficients The following conventions for denoting coefficients of formal power series will be used throughout the whole paper.

1^o For $f \in \Theta_n$ and $k \geq 1$, $1 \leq i_1, \ldots, i_k \leq n$, we will denote

$$[\text{coef } (i_1, \ldots, i_k)](f) \ \stackrel{def}{=} \ \text{the coefficient of } z_{i_1} \cdots z_{i_k} \text{ in } f. \tag{3.2}$$

2^o Restrictions of k-tuples: let $k \geq 1$ and $1 \leq i_1, \ldots, i_k \leq n$ be integers, and let $B = \{h_1 < h_2 < \cdots < h_r\}$ be a non-void subset of $\{1, \ldots, k\}$. Then by "$(i_1, \ldots, i_k)|B$" we will understand the r-tuple $(i_{h_1}, i_{h_2}, \ldots, i_{h_r})$. An expression like

$$[\text{coef } (i_1, \ldots, i_k)|B](f) \tag{3.3}$$

for $f \in \Theta_n$, $k \geq 1$, $1 \leq i_1, \ldots, i_k \leq n$ and $\emptyset \neq B \subseteq \{1, \ldots, k\}$, will hence mean that the convention of notation (3.2) is applied to the $|B|$-tuple $(i_1, \ldots, i_k)|B$.

3^o Given $f \in \Theta_n$, $k \geq 1$, $1 \leq i_1, \ldots, i_k \leq n$ integers, and π a non-crossing partition of $\{1, \ldots, k\}$, we will denote

$$[\text{coef } (i_1, \ldots, i_k); \pi](f) \ \stackrel{def}{=} \prod_{B \text{ block of } \pi} [\text{coef } (i_1, \ldots, i_k)|B](f). \tag{3.4}$$

Thus if n, k, $1 \leq i_1, \ldots, i_k \leq n$ and π are fixed, then $[\text{coef } (i_1, \ldots, i_k); \pi] : \Theta_n \to \mathbf{C}$ is a functional, generally non-linear (it is linear if and only if π is the partition into only one block, $\{\{1, \ldots, k\}\}$, in which case $[\text{coef } (i_1, \ldots, i_k); \pi] = [\text{coef } (i_1, \ldots, i_k)]$ of (3.2)).

4^o The 1-dimensional case: If $n = 1$, then the convention of notation in (3.2) can (and will) be abridged from $[\text{coef } (\underbrace{1, \ldots, 1}_{k})](f)$ to just $[\text{coef } (k)](f)$. A similar abbreviation will

be used for the convention in (3.4); note that the restriction of k-tuples gets in this case the form "$(k)|B = (|B|)$", for $\emptyset \neq B \subseteq \{1, \ldots, k\}$, hence (3.4) becomes:

$$[\mathrm{coef}\ (k); \pi](f) \ = \prod_{B \ block \ of \ \pi} [\mathrm{coef}\ (|B|)](f), \tag{3.5}$$

for $f \in \Theta_1,\ k \geq 1$ and $\pi \in NC(k)$.

3.3 The operation $\boxed{\star}$ Let n be a positive integer. We denote by $\boxed{\star}$ $(= \boxed{\star}_n)$ the binary operation on the set Θ_n of 3.1, determined by the formula

$$[\mathrm{coef}\ (i_1, \ldots, i_k)](f \boxed{\star} g) \ = \sum_{\pi \in NC(k)} [\mathrm{coef}\ (i_1, \ldots, i_k); \pi](f) \cdot [\mathrm{coef}\ (i_1, \ldots, i_k); K(\pi)](g), \tag{3.6}$$

holding for every $f, g \in \Theta_n,\ k \geq 1,\ 1 \leq i_1, \ldots, i_k \leq n$, and where $K : NC(k) \to NC(k)$ is the Kreweras complementation map reviewed in Section 2.3.

The operation $\boxed{\star}$ on Θ_n is associative (see [10], Proposition 3.5) and is commutative when (and only when) $n = 1$ (see e.g. [9], Proposition 1.4.2). The unit for $\boxed{\star}$ is the series which takes the sum of the variables, $Sum(z_1, \ldots, z_n) \stackrel{def}{=} z_1 + \cdots + z_n$.

An important role in the considerations related to $\boxed{\star}$ is played by the series

$$Zeta(z_1, \ldots, z_n) \ = \sum_{k=1}^{\infty} \sum_{i_1, \ldots, i_k = 1}^{n} z_{i_1} \cdots z_{i_k} \tag{3.7}$$

and

$$Moeb(z_1, \ldots, z_n) \ = \sum_{k=1}^{\infty} \sum_{i_1, \ldots, i_k = 1}^{n} (-1)^{k+1} \frac{(2k-2)!}{(k-1)!k!} \, z_{i_1} \cdots z_{i_k}, \tag{3.8}$$

which are called the (n-variable) Zeta and Moebius series, respectively. (The names are coming from the combinatorial interpretation of these series. The relation with the Moebius inversion theory in a poset, as developed in [3], is particularly clear in the case when $n = 1$ - see [9].) $Zeta$ and $Moeb$ are inverse to each other with respect to $\boxed{\star}$, i.e. $Zeta \boxed{\star} Moeb = Sum = Moeb \boxed{\star} Zeta$. Moreover, they are central with respect to $\boxed{\star}$ (i.e. $Zeta \boxed{\star} f = f \boxed{\star} Zeta$ for every $f \in \Theta_n$, and similarly for $Moeb$).

Although chronologically the R-transform preceded the $\boxed{\star}$-operation, it is convenient to define it here in the following way.

3.4 Definition Let n be a positive integer. To every functional $\mu : \mathbf{C}\langle X_1, \ldots, X_n \rangle \to \mathbf{C}$, normalized by $\mu(1) = 1$ (where $\mathbf{C}\langle X_1, \ldots, X_n \rangle$ is the algebra of polynomials in n non-commuting indeterminates, as in (1.2)), we attach two formal power series $M(\mu), R(\mu) \in \Theta_n$,

by the equations:

$$[M(\mu)](z_1, \ldots, z_n) \;=\; \sum_{k=1}^{\infty} \sum_{i_1,\ldots,i_k=1}^{n} \mu(X_{i_1} \cdots X_{i_k}) z_{i_1} \cdots z_{i_k} \tag{3.9}$$

and

$$R(\mu) \;=\; M(\mu) \boxed{\star} Moeb. \tag{3.10}$$

The series $R(\mu)$ is called the R-transform of μ.

3.5 The moment-cumulant formula Since $Moeb$ is the inverse of $Zeta$ under $\boxed{\star}$, the Equation (3.10) is clearly equivalent to

$$M(\mu) \;=\; R(\mu) \boxed{\star} Zeta; \tag{3.11}$$

(3.11) is in some sense "the formula for the R^{-1}-transform", since the transition from $M(\mu)$ back to μ is trivial. The formula (3.11) is more frequently used than (3.10), for the reason that the coefficients of $Zeta$ are easier to handle than those of $Moeb$. The equation obtained by plugging (3.6) into (3.11) and taking into account that $[\text{coef } (i_1, \ldots, i_k); \pi](Zeta)$ is always equal to 1 was first observed in [14], and looks like this:

$$[\text{coef } (i_1, \ldots, i_k)](M(\mu)) \;=\; \sum_{\pi \in NC(k)} [\text{coef } (i_1, \ldots, i_k); \pi](R(\mu)), \tag{3.12}$$

for $k \geq 1$ and $1 \leq i_1, \ldots, i_k \leq n$. We will call (3.12) the *moment-cumulant formula,* because it connects the coefficient $[\text{coef } (i_1, \ldots, i_k)](M(\mu))$ - which is just $\mu(X_{i_1} \cdots X_{i_k})$, a "moment" of μ, with the coefficients of $R(\mu)$ - which were called in [14] the "free (or non-crossing) cumulants" of μ.

The multi-variable R-transform has the fundamental property that it stores the information about a joint distribution (in the sense of Eqn.(1.2)) in such a way that the freeness or non-freeness of the non-commutative random variables involved becomes very transparent. More precisely, we have the following

3.6 Theorem [14,8]: The families of elements $\{a_{1,1}, \ldots, a_{1,m_1}\}, \ldots, \{a_{n,1}, \ldots, a_{n,m_n}\}$ are free in the non-commutative probability space $(\mathcal{A}, \varphi)$ if and only if the coefficient of $z_{i_1,j_1} z_{i_2,j_2} \cdots z_{i_k,j_k}$ in $[R(\mu_{a_{1,1},\ldots,a_{1,m_1},\ldots,a_{n,1},\ldots,a_{n,m_n}})]$ $(z_{1,1}, \ldots, z_{1,m_1}, \ldots, z_{n,1}, \ldots, z_{n,m_n})$ vanishes whenever we don't have $i_1 = i_2 = \cdots = i_k$; i.e. if and only if $R(\mu_{a_{1,1},\ldots,a_{1,m_1},\ldots,a_{n,1},\ldots,a_{n,m_n}})$ is of the form

$$f_1(z_{1,1}, \ldots, z_{1,m_1}) + \cdots + f_n(z_{n,1}, \ldots, z_{n,m_n}) \tag{3.13}$$

for some formal power series $f_1, \ldots, f_n$.

We will refer to the coefficient-vanishing condition in Theorem 3.6 by saying that "the R-series has no mixed coefficients". If this happens, then $f_1, \ldots, f_n$ of (3.13) can only be the R-transforms $R(\mu_{a_{1,1},\ldots,a_{1,m_1}})$, ... , $R(\mu_{a_{n,1},\ldots,a_{n,m_n}})$, respectively.

The Equation (3.6) of 3.3 will be called in the sequel "the combinatorial definition of $\boxed{\star}$". In an approach where the R-transform is defined first, another possible definition of $\boxed{\star}$ would be provided by the following fact (which comes here as a theorem).

3.7 Theorem ([10]) Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and let $a_1, \ldots,$ $a_n, b_1, \ldots b_n \in \mathcal{A}$ be such that $\{a_1, \ldots, a_n\}$ is free from $\{b_1, \ldots, b_n\}$. Then

$$R(\mu_{a_1 b_1, \ldots, a_n b_n}) \;=\; R(\mu_{a_1, \ldots, a_n}) \boxed{\star} R(\mu_{b_1, \ldots, b_n}). \tag{3.14}$$

We mention that it is often handier to use the Equation (3.14) after performing a $\boxed{\star}$-operation with $Zeta$ on the right on both sides, and invoking (3.11); this leads to the equivalent equations:

$$M(\mu_{a_1 b_1, \ldots, a_n b_n}) \;=\; R(\mu_{a_1, \ldots, a_n}) \boxed{\star} M(\mu_{b_1, \ldots, b_n}) \tag{3.15}$$

or

$$M(\mu_{a_1 b_1, \ldots, a_n b_n}) \;=\; M(\mu_{a_1, \ldots, a_n}) \boxed{\star} R(\mu_{b_1, \ldots, b_n}) \tag{3.16}$$

(obtained by attaching the $Zeta$ on the right-hand side to $R(\mu_{b_1,\ldots,b_n})$ and to $R(\mu_{a_1,\ldots,a_n})$, respectively).

Finally, since in this paper we are dealing only with tracial non-commutative probability spaces, it is useful to recall the following simple fact concerning cyclic permutations in the R-transform.

3.8 Proposition Let n be a positive integer and let $\mu : \mathbf{C}\langle X_1, \ldots, X_n\rangle \to \mathbf{C}$ be a linear functional, such that $\mu(1) = 1$. Assume that μ is a trace (i.e. $\mu(p'p'') = \mu(p''p')$ for every two polynomials $p', p'' \in \mathbf{C}\langle X_1, \ldots, X_n\rangle$; this is for instance the case whenever $\mu = \mu_{a_1,\ldots,a_n}$ for some elements $a_1, \ldots, a_n$ in some $(\mathcal{A}, \varphi)$, with φ a trace). Then the coefficients of $R(\mu)$

are invariant under cyclic permutations, i.e.

$$[\text{coef } (i_1, \ldots, i_k, j_1, \ldots, j_l)](R(\mu)) \;=\; [\text{coef } (j_1, \ldots, j_l, i_1, \ldots, i_k)](R(\mu)) \tag{3.17}$$

for every $k, l \geq 1$ and $1 \leq i_1, \ldots, i_k, j_1, \ldots, j_l \leq n$.

The proof of Proposition 3.8 comes out immediately by using the moment-cumulant formula, see e.g. [15], Section 2.4.

4. A basic combinatorial remark

4.1 Definition Let n be a positive integer, let σ be in $NC(2n)$, and consider the permutation $Perm(\sigma) \in \mathcal{S}_{2n}$ associated to σ as in Section 2.6. We will say that σ is *parity-alternating* (respectively *parity-preserving*) if the difference $i - [Perm(\sigma)](i) \in \mathbf{Z}$ is odd (respectively even) for every $1 \leq i \leq 2n$. We will use the notations

$$\left\{ \begin{array}{lcl} \{\sigma \in NC(2n) \mid \sigma \text{ parity-alternating } \} & \overset{def}{=} & NC_{p-alt}(2n), \\ \{\sigma \in NC(2n) \mid \sigma \text{ parity-preserving}\} & \overset{def}{=} & NC_{p-prsv}(2n). \end{array} \right. \tag{4.1}$$

4.2 Remark It is clear that a partition $\sigma \in NC(2n)$ is parity-preserving if and only if each of its blocks either is contained in $\{1, 3, \ldots, 2n-1\}$ or is contained in $\{2, 4, \ldots, 2n\}$. For parity-alternating partitions, an equivalent description is:

$$\sigma \in NC_{p-alt}(2n) \;\Leftrightarrow\; \{ \text{ every block of } \sigma \text{ has an even number of elements}\}. \tag{4.2}$$

Implication "$\Rightarrow$" in (4.2) is immediate, while "$\Leftarrow$" uses the fact that if B is a block of σ and if $i < j$ in B are such that $\{i+1, \ldots, j-1\} \cap B = \emptyset$, then the interval $\{i+1, \ldots, j-1\}$ is a union of other blocks of σ (hence the named interval has an even number of elements, and hence i and j have different parities).

4.3 Remark The Kreweras complementation map $K : NC(2n) \to NC(2n)$ puts in bijection parity-alternating and parity-preserving partitions; we have in fact both equalities:

$$\left\{ \begin{array}{lcl} K(NC_{p-alt}(2n)) & = & NC_{p-prsv}(2n), \\ K(NC_{p-prsv}(2n)) & = & NC_{p-alt}(2n). \end{array} \right. \tag{4.3}$$

Indeed, denoting the cycle $(1 \to 2 \to \cdots \to 2n \to 1)$ by γ, we know (Eqn.(2.4)) that $Perm(K(\sigma)) = Perm(\sigma)^{-1}\gamma$, $\sigma \in NC(2n)$; this equality and the fact that γ itself is parity-alternating imply together the inclusion "$\subseteq$" of both Equations (4.3). But then, since K is one-to-one, we get that both inequalities between $|NC_{p-alt}(2n)|$ and $|NC_{p-prsv}(2n)|$ are holding; hence $|NC_{p-alt}(2n)| = |NC_{p-prsv}(2n)|$, and in (4.3) we must really have equalities.

The combinatorial remark mentioned in the title of the section is the following.

4.4 Proposition Let n be a positive integer. In order to distinguish the notations, we will write $K^{(2n)}$ for the Kreweras complementation map on $NC(2n)$ (while the Kreweras map on $NC(n)$ will be simply denoted by K).

1^o We have a canonical bijection

$$\{(\pi, \rho) \mid \pi, \rho \in NC(n),\ \pi \leq \rho\ \} \ni (\pi, \rho) \ \longrightarrow\ \sigma \in NC_{p-prsv}(2n), \tag{4.4}$$

where σ of (4.4) is determined as follows: the restriction $\sigma|\{2, 4, \ldots, 2n\}$ is obtained by transporting π from $\{1, \ldots, n\}$ to $\{2, 4, \ldots, 2n\}$, while the restriction $\sigma|\{1, 3, \ldots, 2n-1\}$ is obtained by transporting $K^{-1}(\rho)$ from $\{1, \ldots, n\}$ to $\{1, 3, \ldots, 2n-1\}$. (In terms of equivalence relations associated to partitions, we have $2i \overset{\sigma}{\sim} 2j \Leftrightarrow i \overset{\pi}{\sim} j$ and $2i-1 \overset{\sigma}{\sim} 2j-1 \Leftrightarrow i \overset{K^{-1}(\rho)}{\sim} j$, for every $1 \leq i, j \leq n$.)

2^o If $\pi \leq \rho$ in $NC(n)$ and $\sigma \in NC_{p-prsv}(2n)$ are as above, then the relative Kreweras complement $K_\rho(\pi) \in NC(n)$ (described in Section 2.5) is given by the formula:

$$K_\rho(\pi) \ = \ \left\{ \ \{\frac{b}{2} \mid b \in B \cap \{2, 4, \ldots, 2n\}\} \ ; \ B \text{ block of } K^{(2n)}(\sigma) \right\}. \tag{4.5}$$

Proof 1^o is an immediate consequence of Proposition 2.4.

For 2^o, we will compare the permutations associated to the two partitions appearing in (4.5) (the right-hand side of (4.5), which could be written as $\frac{1}{2}[K^{(2n)}(\sigma)|\{2, 4, \ldots, 2n\}]$, is also in $NC(n)$; this is because the naturally defined operation of restriction preserves the quality of a partition of being non-crossing - easy verification).

Let us denote the permutations $Perm(\pi)$ and $Perm(\rho)$ by α and β, respectively. Then we know (Eqn.(2.5)) that $Perm(K_\rho(\pi)) = \alpha^{-1}\beta$. On the other hand, let us denote the partition in the right-hand side of (4.5) by θ. We have that $K^{(2n)}(\sigma) \in NC_{p-alt}(2n)$ (because $\sigma \in NC_{p-prsv}(2n)$ and by Remark 4.3), and this immediately implies the following formula,

expressing $Perm(\theta)$ in terms of $Perm(K^{(2n)}(\sigma))$:

$$2[Perm(\theta)](i) \ = \ \Big(Perm(K^{(2n)}(\sigma))\Big)^2 (2i), \quad 1 \leq i \leq n. \tag{4.6}$$

Hence what we have to do is calculate $Perm(K^{(2n)}(\sigma))$ in terms of α and β, and verify that the right-hand side of (4.6) equals $2\alpha^{-1}(\beta(i))$.

Now, the definition of σ in terms of π and ρ is converted at the level of the associated permutations by saying that $Perm(\sigma) \in \mathcal{S}_{2n}$ acts by:

$$\begin{cases} [Perm(\sigma)](2i) \ = \ 2[Perm(\pi)](i) \ = \ 2\alpha(i), \\ [Perm(\sigma)](2i-1) \ = \ 2[Perm(K^{-1}(\rho))](i) - 1, \end{cases} \quad 1 \leq i \leq n.$$

It is more convenient to look at $Perm(\sigma)^{-1}$, which is thus given by

$$\begin{cases} [Perm(\sigma)]^{-1}(2i) \ = \ 2\alpha^{-1}(i), \\ [Perm(\sigma)]^{-1}(2i-1) \ = \ 2[Perm(K^{-1}(\rho))]^{-1}(i) - 1, \end{cases} \quad 1 \leq i \leq n. \tag{4.7}$$

Denoting the cycles $(1 \to 2 \to \cdots \to n \to 1) \in \mathcal{S}_n$ and $(1 \to 2 \to \cdots \to 2n \to 1) \in \mathcal{S}_{2n}$ by γ_n and γ_{2n}, respectively, we get (by appropriately substituting in (2.4)) that $Perm(K^{-1}(\rho))^{-1} \ = \ \beta\gamma_n^{-1}$ and that $Perm(K^{(2n)}(\sigma)) \ = \ Perm(\sigma)^{-1}\gamma_{2n}$. Hence for every $1 \leq i \leq n$:

$$[Perm(K^{(2n)}(\sigma))](2i-1) \ = \ [Perm(\sigma)^{-1}\gamma_{2n}](2i-1) \ = \ Perm(\sigma)^{-1}(2i) \ \overset{(4.7)}{=} \ 2\alpha^{-1}(i); \tag{4.8}$$

and for every $1 \leq i \leq n-1$, a similar calculation leads to

$$[Perm(K^{(2n)}(\sigma))](2i) \ = \ 2\beta(i) - 1. \tag{4.9}$$

Equation (4.9) must also hold for $i = n$, because $Perm(K^{(2n)}(\sigma))$ is a bijection. From (4.8) and (4.9) it is easily verified that the right-hand side of (4.6) is indeed $2\alpha^{-1}(\beta(i))$, and this concludes the proof. **QED**

4.5 Corollary There exists a bijection between $\{(\pi, \rho) \mid \pi, \rho \in NC(n),\ \pi \leq \rho\}$ and $NC_{p-alt}(2n)$, such that, for (π, ρ) in the first set corresponding to τ in the second:

(i) π and τ have the same number, say k, of blocks,

and moreover,

(ii) we can write $\pi = \{A_1, \ldots, A_k\}$ and $\tau = \{B_1, \ldots, B_k\}$ in such a way that $|A_j| = \frac{1}{2}|B_j|$ for every $1 \leq j \leq k$.

(Note: The less involved fact that $\{(\pi, \rho) \mid \pi, \rho \in NC(n),\ \pi \leq \rho\}$ and $NC_{p-alt}(2n)$ have the same number - namely $(3n)!/n!(2n+1)!$ - of elements, was well-known, see e.g. [6] or Section 2 of [14].)

Proof The desired bijection is the diagonal arrow of the diagram

$$\begin{tikzcd}
\{(\pi, \rho) \mid \pi, \rho \in NC(n),\ \pi \leq \rho\} \arrow[r, dashed] \arrow[d, dashed] & NC_{p-prsv}(2n) \arrow[r, dashed, "K^{(2n)}"] & NC_{p-alt}(2n) \\
\{(\pi, \rho) \mid \pi, \rho \in NC(n),\ \pi \leq \rho\} & &
\end{tikzcd} \tag{4.10}$$

where the leftmost horizontal arrow is the bijection described in Proposition $4.4.1^o$, and the vertical arrow is the bijection $(\pi, \rho) \rightarrow (K_\rho(\pi), \rho)$. Indeed, if $(\pi, \rho) \rightarrow \tau$ by the diagonal arrow of (4.10), then from Proposition $4.4.2^o$ it follows that $\pi = \Big\{ \ \{\frac{b}{2} \mid b \in B \cap \{2, 4, \ldots, 2n\}\}$; B block of $\tau\}$; but since τ is parity-alternating, it is clear that $|\{B \cap \{2, 4, \ldots, 2n\}| = |B|/2$ for every block B of τ. **QED**

We now pass to the proof of the Theorem 1.14. We use the same line as for the mentioned Application 1.10 of [10], which relies on the following

4.6 Freeness Criterion: Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, such that φ is a trace, let $\mathcal{C} \subseteq \mathcal{A}$ be a unital subalgebra, and let $\mathcal{X} \subseteq \mathcal{A}$ be a non-void subset. Assume that for every $m \geq 1$ and $c_1, \ldots, c_m \in \mathcal{C}$, $x_1, \ldots, x_m \in \mathcal{X}$, it is true that

$$\varphi(c_1 x_1 c_2 x_2 \cdots c_m x_m) \ = \ [\text{coef } (1, \ldots, m)](R(\mu_{c_1, \ldots, c_m}) \boxed{\star} M(\mu_{x_1, \ldots, x_m})) \tag{4.11}$$

(where, according to the notations set in Section 3.2, the right-hand side of (4.11) denotes the coefficient of $z_1 \cdots z_m$ in the formal power series $R(\mu_{c_1, \ldots, c_m}) \boxed{\star} M(\mu_{x_1, \ldots, x_m})$). Then $\mathcal{X}$ is free from $\mathcal{C}$ in $(\mathcal{A}, \varphi)$.

For the proof of 4.6, see [10], Proposition 4.7. In addition to this criterion, we will use the following simple lemma.

4.7 Lemma Given $m \geq 1$ and $\pi \in NC(m)$, the following are equivalent:

1^o there exists a block B of π such that $|B|$ is odd;

2^o there exists a block $B = \{i_1 < i_2 < \cdots < i_k\}$ of π such that either $k = 1$, or $k \geq 3$ is odd and all the differences $i_2 - i_1, i_3 - i_2, \ldots, i_k - i_{k-1}$ are also odd.

Proof We only need to show that $1^o \Rightarrow 2^o$. We proceed by induction on m. The case $m = 1$ is clear, we prove the induction step $\{1, \ldots, m-1\} \Rightarrow m$. We fix $\pi \in NC(m)$ which satisfies 1^o, and we also fix a block B of π such that $|B|$ is odd. Clearly, we can assume that $|B| \neq 1$ and that there exist two elements $i < j$ in B such that $\{i+1, \ldots, j-1\} \cap B = \emptyset$ and such that $j - i$ is even (otherwise we are done). We also fix i and j. The interval $\{i+1, \ldots, j-1\}$ is a union of blocks of B; let $\pi_o \in NC(j-i-1)$ be the partition obtained from $\pi|\{i+1, \ldots, j-1\}$ by translation with i downwards. Then π_o satisfies 1^o (because $j - i - 1$ is odd), hence also 2^o (by the induction hypothesis). It only remains to pick a block of π_o which satisfies 2^o, and shift it back (with i upwards) into a block of π. **QED**

Proof of Theorem 1.14 Let $(\mathcal{A}, \varphi)$ and $a_1, \ldots, a_n, b', b'' \in \mathcal{A}$ be as in the statement of the theorem. Let $\mathcal{C}$ denote the unital subalgebra of $\mathcal{A}$ generated by $a_1, \ldots, a_n$, and put $\mathcal{X} = \{b'a_1b'', \ldots, b'a_nb''\}$. The sufficient freeness condition of 4.6 becomes here:

$$\varphi(c_1(b'a_{i_1}b'')c_2(b'a_{i_2}b'') \cdots c_m(b'a_{i_m}b'')) \tag{4.12}$$

$$= [\text{coef } (1, \ldots, m)] \left(R(\mu_{c_1,\ldots,c_m}) \boxed{\star} M(\mu_{b'a_{i_1}b'',\ldots,b'a_{i_m}b''}) \right),$$

to be verified for every $m \geq 1$, $c_1, \ldots, c_m \in \mathcal{C}$ and $1 \leq i_1, \ldots, i_m \leq n$.

We start with the left-hand side of (4.12), and rewrite it as

$$\varphi((c_1b')(a_{i_1}b'')(c_2b')(a_{i_2}b'') \cdots (c_mb')(a_{i_m}b''))$$

$$= [\text{coef } (1, 2, \ldots, 2m)](M(\mu_{c_1b',a_{i_1}b'',\ldots,c_mb',a_{i_m}b''})$$

$$= [\text{coef } (1, 2, \ldots, 2m)](R(\mu_{c_1,a_{i_1},\ldots,c_m,a_{i_m}}) \boxed{\star} M(\mu_{\underbrace{b', b'', \ldots, b', b''}_{2m}})),$$

(because $\{b', b''\}$ is free from $\{c_1, \ldots, c_m, a_{i_1}, \ldots, a_{i_m}\}$ and by the formula (3.15))

$$= \sum_{\sigma \in NC(2m)} [\text{coef } (1, 2, \ldots, 2m); \sigma](R(\mu_{c_1,a_{i_1},\ldots,c_m,a_{i_m}})) \cdot \tag{4.13}$$

$$[\text{coef } (1, 2, \ldots, 2m); K^{(2m)}(\sigma)](M(\mu_{b',b'',\ldots,b',b''}))$$

(by the combinatorial definition of $\boxed{\star}$ in 3.3).

Now, let us remark that the quantity [coef $(1,2,\ldots,2m); K^{(2m)}(\sigma)](M(\mu_{b',b'',\ldots,b',b''}))$ appearing in (4.13) vanishes whenever $K^{(2m)}(\sigma)$ is not parity-alternating. Indeed, if $K^{(2m)}(\sigma) \notin NC_{p-alt}(2m)$, then $K^{(2m)}(\sigma)$ must have at least one block B with $|B|$ odd (by Remark 4.2), hence also at least one block B with the property stated in 4.7.2^o (by the Lemma 4.7). But for any block B with the latter property we have [coef $(1,2,\ldots,2m)|B](M(\mu_{b',b'',\ldots,b',b''}))$ $=0$, because (b',b'') is diagonally balanced; this implies that [coef $(1,2,\ldots,2m); K^{(2m)}(\sigma)]$ $(M(\mu_{b',b'',\ldots,b',b''}))$ $=0$, by Eqn.(3.4).

By also invoking the Remark 4.3, we hence see that in (4.13) we can in fact sum only over $\sigma \in NC_{p-prsv}(2m)$ (the contribution of a $\sigma \notin NC_{p-prsv}(2m)$ is always zero).

But then we can perform in (4.13) the "substitution" indicated by the bijection of Proposition 4.4.1^o, i.e. we can pass to a sum indexed by $\{(\pi,\rho) \mid \pi,\rho \in NC(m),\, \pi \leq \rho\}$. Of course, in order to do this we need to rewrite the general summand of (4.13) not in terms of $\sigma \in NC_{p-prsv}(2m)$, but in terms of the pair (π,ρ) corresponding to it. We leave it to the reader to verify that

(a) the part [coef $(1,2,\ldots,2m);\sigma](R(\mu_{c_1,a_{i_1},\ldots,c_m,a_{i_m}}))$ of the general summand gets converted into the product

$$[\text{coef } (1,\ldots,m);\pi](R(\mu_{a_{i_1},\ldots,a_{i_m}})) \cdot [\text{coef } (1,\ldots,m);K^{-1}(\rho)](R(\mu_{c_1,\ldots,c_m}));$$

and

(b) due to the description of $K_\rho(\pi)$ in Proposition 4.4.2^o, the part [coef $(1,2,\ldots,2m)$; $K^{(2m)}(\sigma)]$ $(M(\mu_{\underbrace{b',b'',\ldots,b',b''}_{2m}}))$ of the general summand gets converted into the coefficient [coef $(1,\ldots,m); K_\rho(\pi)](M(\mu_{\underbrace{b''b',\ldots,b''b'}_{m}}))$. (Indeed, if we write $K^{(2m)}(\sigma) = \{B_1,\ldots,B_k\}$ and $K_\rho(\pi) = \{A_1,\ldots,A_k\}$ in such a way that $|B_j| = 2|A_j|$, $1 \leq j \leq k$, then the both [coef $\ldots$] quantities claimed to be equal are seen to be just $\prod_{j=1}^k \varphi((b''b')^{|A_j|})$. We are of course using here the fact that $\varphi((b'b'')^l) = \varphi((b''b')^l)$ for every $l \geq 0$, which holds because φ is a trace.)

The conclusion of (a) and (b) above is that (4.13) can be continued with:

$$\sum_{\substack{\pi,\rho\in NC(n)\\ such\ that\\ \pi\leq\rho}} [\text{coef } (1,\ldots,m);K^{-1}(\rho)](R(\mu_{c_1,\ldots,c_m})) \cdot [\text{coef } (1,\ldots,m);\pi](R(\mu_{a_{i_1},\ldots,a_{i_m}}))\cdot$$
$$[\text{coef } (1,\ldots,m);K_\rho(\pi)](M(\mu_{b''b',\ldots,b''b'})). \tag{4.14}$$

It is not hard to check that the quantity in (4.14) is exactly

$$[\text{coef } (1, \ldots, m)](R(\mu_{c_1,\ldots,c_m}) \boxed{\star} R(\mu_{a_{i_1},\ldots,a_{i_m}}) \boxed{\star} M(\mu_{b''b',\ldots,b''b'})) \tag{4.15}$$

(indeed, one has just to look at the combinatorial formula for the $\boxed{\star}$-product of three series f, g, h, calculated in the order $f \boxed{\star} (g \boxed{\star} h)$) - compare for instance to [10], Eqn.(3.7)).

Finally, since $b''b'$ is free from $\{a_{i_1}, \ldots, a_{i_m}\}$ we have (by applying again (3.15)) that $R(\mu_{a_{i_1},\ldots,a_{i_m}}) \boxed{\star} M(\mu_{b''b',\ldots,b''b'}) = M(\mu_{a_{i_1}b''b',\ldots,a_{i_m}b''b'})$; the latter series can also be written as $M(\mu_{b'a_{i_1}b'',\ldots,b'a_{i_m}b''})$, because we are working in a tracial non-commutative probability space. We thus arrive to the fact that (4.15) coincides with the right-hand side of (4.12), and this concludes the proof. **QED**

5. Diagonally balanced pairs

We first show that Definition 1.12 could have been equally well formulated in terms of the free cumulants of a_1 and a_2 (rather than their moments).

5.1 Proposition Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and let a_1, a_2 be in $\mathcal{A}$. The pair (a_1, a_2) is diagonally balanced if and only if the coefficients of $\underbrace{z_1 z_2 \cdots z_1 z_2 z_1}_{2n+1}$ and $\underbrace{z_2 z_1 \cdots z_2 z_1 z_2}_{2n+1}$ in $[R(\mu_{a_1,a_2})](z_1, z_2)$ are equal to zero for every $n \geq 0$.

Proof This is an immediate consequence of the Lemma 4.7 and the Equations (3.10), (3.11) connecting the M- and R-series of μ_{a_1,a_2}. For instance, when proving the "$\Rightarrow$" part, we write

$$[\text{coef } (1, 2, \ldots, 1, 2, 1)](R(\mu_{a_1,a_2})) = [\text{coef } (1, 2, \ldots, 1, 2, 1)](M(\mu_{a_1,a_2}) \boxed{\star} Moeb) \tag{5.1}$$

$$= \sum_{\pi \in NC(2n+1)} [\text{coef } (1, 2, \ldots, 1, 2, 1); \pi](M(\mu_{a_1,a_2})) \cdot [\text{coef } (1, 2, \ldots, 1, 2, 1); K(\pi)](Moeb)$$

(and a similar formula for $[\text{coef } (2, 1, \ldots, 2, 1, 2)](R(\mu_{a_1,a_2}))$). Then we note that in the sum (5.1) every term is actually vanishing; indeed, the Lemma 4.7 and the hypothesis that (a_1, a_2) is a diagonally balanced pair imply together that $[\text{coef } (1, 2, \ldots, 1, 2, 1); \pi](M(\mu_{a_1,a_2}))$ $= 0$ for every $\pi \in NC(2n+1)$. **QED**

5.2 Remark It is useful to record here that if (a_1, a_2) is a diagonally balanced pair in $(\mathcal{A}, \varphi)$, where φ is a trace, then the coefficient of $z_{i_1} z_{i_2} \cdots z_{i_{2n+1}}$ in $[R(\mu_{a_1,a_2})](z_1, z_2)$ is also vanishing whenever $(i_1, i_2, \ldots, i_{2n+1})$ is a cyclic permutation of $(1, 2, \ldots, 1, 2, 1)$ or of $(2, 1, \ldots, 2, 1, 2)$. This follows from Propositions 5.1 and 3.8.

Since (as it is clear from 1.3 and 5.1) every R-diagonal pair is diagonally balanced, the next result contains the Proposition 1.7 as a particular case.

5.3 Proposition Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space such that φ is a trace, and let (a_1, a_2) be a diagonally balanced pair in $(\mathcal{A}, \varphi)$. For every $k \geq 1$ we denote by α_k the coefficient of $(z_1 z_2)^k$ (equivalently, of $(z_2 z_1)^k$) in the series $[R(\mu_{a_1,a_2})](z_1, z_2)$, and we consider the series of one variable $f(z) = \sum_{k=1}^{\infty} \alpha_k z^k$. Then $f = R(\mu_{a_1 a_2}) \boxed{\star} Moeb$, where $Moeb$ is the Moebius series, as in Eqn.(1.9) (and of course $\mu_{a_1 a_2} : \mathbf{C}[X] \to \mathbf{C}$ denotes the distribution of the product $a_1 \cdot a_2$).

Proof For every $n \geq 1$ we have:

$$\begin{aligned} \varphi((a_1 a_2)^n) &= [\text{coef } (\underbrace{1, 2, \ldots, 1, 2}_{2n})](M(\mu_{a_1,a_2})) \\ &\overset{(3.12)}{=} \sum_{\tau \in NC(2n)} [\text{coef } (\underbrace{1, 2, \ldots, 1, 2}_{2n}); \tau](R(\mu_{a_1,a_2})). \end{aligned} \tag{5.2}$$

If the partition $\tau \in NC(2n)$ appearing in (5.2) is not in $NC_{p-alt}(2n)$, then it has at least one block B with $|B|$ odd (by Remark 4.2), hence it also has a block B satisfying condition 2^o of Lemma 4.7. This and the description of the diagonally balanced pair (a_1, a_2) in terms of cumulants (Proposition 5.1) imply together that $[\text{coef } (\underbrace{1, 2, \ldots, 1, 2}_{2n}); \tau](R(\mu_{a_1,a_2})) = 0$. On the other hand, if $\tau = \{B_1, \ldots, B_h\}$ is in $NC_{p-alt}(2n)$, then by the very definition of $NC_{p-alt}(2n)$ we have that

$$[\text{coef } (\underbrace{1, 2, \ldots, 1, 2}_{2n}); \tau](R(\mu_{a_1,a_2})) = \alpha_{|B_1|/2} \cdots \alpha_{|B_h|/2}$$

(where the α's are as in the statement of the proposition). This argument shows that the sum in (5.2) is equal to:

$$\sum_{\substack{\tau \in NC_{p-alt}(2n) \\ \tau = \{B_1, \ldots, B_h\}}} \alpha_{|B_1|/2} \cdots \alpha_{|B_h|/2}. \tag{5.3}$$

We next transform the sum in (5.3) by using the bijection between $NC_{p-alt}(2n)$ and $\{(\pi,\rho) \mid \pi,\rho \in NC(n),\ \pi \leq \rho\}$ indicated in Corollary 4.5; we obtain:

$$\sum_{\substack{\pi,\rho \in NC(n) \\ such\ that\ \pi \leq \rho, \\ \pi = \{A_1,\ldots,A_h\}}} \alpha_{|A_1|} \cdots \alpha_{|A_h|}$$

$$= \sum_{\substack{\pi \in NC(n) \\ \pi = \{A_1,\ldots,A_h\}}} \alpha_{|A_1|} \cdots \alpha_{|A_h|} \cdot \text{ card } \{\rho \in NC(n) \mid \rho \geq \pi\}. \tag{5.4}$$

Now, for a given $\pi = \{A_1, \ldots, A_h\} \in NC(n)$, the product $\alpha_{|A_1|} \cdots \alpha_{|A_h|}$ is just $[\text{coef } (n); \pi](f)$ (in the sense of the notations in $3.2.4^o$), while on the other hand it is an easy exercise to identify card $\{\rho \in NC(n) \mid \rho \geq \pi\}$ as $[\text{coef } (n); K(\pi)](Zeta \boxed{\star} Zeta)$. Hence (5.4) can be continued with:

$$\sum_{\pi \in NC(n)} [\text{coef } (n); \pi](f) \cdot [\text{coef } (n); K(\pi)](Zeta \boxed{\star} Zeta) \ = \ [\text{coef } (n)](f \boxed{\star} Zeta \boxed{\star} Zeta).$$

But the expression we had started with was $\varphi((a_1 a_2)^n)$, i.e. $[\text{coef } (n)](M(\mu_{a_1 \cdot a_2}))$. We have in other words proved the equality

$$M(\mu_{a_1 \cdot a_2}) \ = \ f \boxed{\star} Zeta \boxed{\star} Zeta \tag{5.5}$$

(since the two series have identical coefficients). It only remains to take the $\boxed{\star}$ product with $Moeb \boxed{\star} Moeb$ on both sides of (5.5), and take into account that $Moeb$ is the inverse of $Zeta$ under $\boxed{\star}$, and that $M(\mu_{a_1 \cdot a_2}) \boxed{\star} Moeb = R(\mu_{a_1 \cdot a_2})$. **QED**

6. The line of proof for the Theorem 1.5

We begin by proving the particular case when the pair $\{p_1, p_2\}$ in Theorem 1.5 is of the form $\{p, 1\}$; that is:

6.1 Proposition Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, such that φ is a trace, and let $a_1, a_2, p \in \mathcal{A}$ be such that (a_1, a_2) is an R-diagonal pair, and p is free from $\{a_1, a_2\}$. Then $(a_1 p, a_2)$ is also an R-diagonal pair.

Proof We fix $m \geq 1$ and a string $\varepsilon = (l_1, \ldots, l_m) \in \{1,2\}^m$ for some $m \geq 1$, such that $[\text{coef } (l_1, \ldots, l_m)](R(\mu_{a_1 p, a_2})) \neq 0$. Our task is to show that m is even, and that the string ε is either $(\underbrace{1,2,1,2,\ldots,1,2}_{m})$ or $(\underbrace{2,1,2,1,\ldots,2,1}_{m})$. (This would verify Condition (ii) in Definition 1.3; note that Condition (i) of 1.3 is automatically verified, since it is assumed that φ is a trace.)

The couples (a_1, a_2) and $(p, 1)$ are free, hence we can use Theorem 3.7 to infer that $R(\mu_{a_1 p, a_2}) = R(\mu_{a_1, a_2}) \boxed{\star} R(\mu_{p,1})$; the combinatorial definition of $\boxed{\star}$ applied to this situation yields then the formula:

$$[\text{coef } (l_1, \ldots, l_m)](R(\mu_{a_1 p, a_2})) \;=\; \sum_{\tau \in NC(m)} [\text{coef } (l_1, \ldots, l_m); \tau](R(\mu_{a_1, a_2})) \cdot \qquad (6.1)$$

$$[\text{coef } (l_1, \ldots, l_m); K(\tau)](R(\mu_{p,1})).$$

Since the left-hand side of (6.1) is assumed to be non-zero, we can choose (and fix) a partition $\tau \in NC(m)$ such that

$$\left\{ \begin{array}{l} [\text{coef } (l_1, \ldots, l_m); \tau](R(\mu_{a_1, a_2})) \neq 0 \\ [\text{coef } (l_1, \ldots, l_m); K(\tau)](R(\mu_{p,1})) \neq 0. \end{array} \right. \qquad (6.2)$$

The first condition (6.2) together with the hypothesis that (a_1, a_2) is an R-diagonal pair imply together that for every block $B = \{i_1 < i_2 < \cdots < i_k\}$ of τ : k is even, and $l_{i_1} \neq l_{i_2}$, $l_{i_2} \neq l_{i_3}, \ldots, l_{i_{k-1}} \neq l_{i_k}$. This already implies that m is even, and that among $l_1, \ldots, l_m$ there are $m/2$ occurrences of 1 and $m/2$ occurrences of 2.

In order to get the interpretation for the second condition (6.2), we note that, since p is always free from 1, we have (by Theorem 3.6, in fact its particular case stated in Eqn.(1.4)):

$$[R(\mu_{p,1})](z_1, z_2) \;=\; [R(\mu_p)](z_1) + [R(\mu_1)](z_2) \;=\; [R(\mu_p)](z_1) + z_2. \qquad (6.3)$$

By taking this into account, we see that if the second condition (6.2) is satisfied, then $\{i\}$ has to be a one-element block of $K(\tau)$ whenever we have $l_i = 2$. A quick look at how the Kreweras complement $K : NC(m) \to NC(m)$ is defined shows that $\{i\}$ is a one-element block of $K(\tau)$ if and only if i and $i+1$ are in the same block of τ (where if $i = m$ we read "1" instead of "$i+1$"). Hence whenever we have $l_i = 2$, we also have that i and $i+1$ lie in the same block of τ, and then from the discussion in the preceding paragraph we deduce that $l_{i+1} = 1$.

We have thus proved that: m is even; among $l_1, \ldots, l_m$ there are $m/2$ of 1 and $m/2$ of 2; and $l_i = 2 \Rightarrow l_{i+1} = 1$, for $1 \leq i \leq m-1$, and $l_m = 2 \Rightarrow l_1 = 1$. We leave it as an elementary

exercise to the reader to verify that a string $(l_1, \ldots, l_m)$ with all these properties can only be $(1, 2, 1, 2, \ldots, 1, 2)$ or $(2, 1, 2, 1, \ldots, 2, 1)$. **QED**

6.2 Remark With exactly the same argument, we could have shown that (a_1, a_2) R-diagonal $\Rightarrow$ (a_1, pa_2) R-diagonal (i.e. the particular case $\{p_1, p_2\} = \{1, p\}$ of Theorem 1.5). Since (obviously) R-diagonality is not affected by reversing the order of the elements of a pair, it follows that in Proposition 6.1 the element p could have been in fact inserted anywhere (left or right of either a_1 or a_2). We take the occasion to mention here that (as the reader can easily verify by experimenting on coefficients of small length) the Theorem 1.5 itself becomes false if in its statement the pair (a_1p_1, p_2a_2) is replaced for instance with (a_1p_1, a_2p_2).

In view of what has been proved up to now, the Theorem 1.5 is equivalent to the following statement.

6.3 Proposition Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, such that φ is a trace, and let $a_1, a_2, p_1, p_2 \in \mathcal{A}$ be such that (a_1, a_2) is an R-diagonal pair, and such that $\{p_1, p_2\}$ is free from $\{a_1, a_2\}$. Then $\mu_{a_1p_1, p_2a_2} = \mu_{a_1p_1p_2, a_2}$.

Indeed, Theorem 1.5 follows from Propositions 6.1 and 6.3, while conversely, Proposition 6.3 is implied by Theorem 1.5 and Corollary 1.8.

Proposition 6.3 simply says that if we put (in the notations used there) $x_1 = a_1p_1, x_2 = p_2a_2$, $y_1 = a_1p_1p_2, y_2 = a_2$, then we have

$$\varphi(x_{l_1}x_{l_2}\cdots x_{l_m}) \ = \ \varphi(y_{l_1}y_{l_2}\cdots y_{l_m}) \tag{6.4}$$

for every $m \geq 1$ and $l_1, \ldots, l_m \in \{1, 2\}$. The point of view we will take concerning (6.4) is the following: when we write back $x_1 = a_1p_1, x_2 = p_2a_2$, $y_1 = a_1p_1p_2, y_2 = a_2$, then both $x_{l_1}x_{l_2}\cdots x_{l_m}$ and $y_{l_1}y_{l_2}\cdots y_{l_m}$ are converted into monomials of length $2m$ in a_1, a_2, p_1, p_2; and moreover, both these monomials of length $2m$ can be regarded as the result obtained by starting with $a_{l_1}a_{l_2}\cdots a_{l_m}$ and inserting p_1's and p_2's in between the a's, according to a certain pattern:

- in order to obtain $x_{l_1}x_{l_2}\cdots x_{l_m}$, we insert a p_1 immediately to the right of each occurrence of a_1, and a p_2 immediately to the left of each occurrence of a_2;

- in order to obtain $y_{l_1}y_{l_2}\cdots y_{l_m}$, we insert a p_1p_2 immediately to the right of each

occurrence of a_1.

So in some sense what we have to do is compare the two insertions patterns described above. Due to its length, this will be done separately in the next section.

We only make now one simple remark: since φ is a trace, both monomials appearing in (6.4) can be permuted cyclically; hence if we assume in (6.4) that $l_1 = 1$, we are in fact missing only the case when $l_1 = l_2 = \cdots = l_m = 2$. But the latter case can be easily settled if we note that $\varphi(b_1^n) = \varphi(b_2^n) = 0$ for every R-diagonal pair (b_1, b_2) and for every $n \geq 1$ (this in turn is an immediate consequence of the moment-cumulant formula (3.12), combined with the particular form of the series $R(\mu_{b_1,b_2})$). Indeed, if $l_1 = l_2 = \cdots = l_m = 2$, then the right-hand side of (6.4) becomes $\varphi(a_2^m) = 0$. In order to verify that the left-hand side of (6.4) is also vanishing, we can invoke the same argument, where we use in addition the Remark 6.2 ($p_2 a_2$ enters the R-diagonal pair $(a_1, p_2 a_2)$, hence $\varphi((p_2 a_2)^m) = 0$).

7. Insertion patterns associated to a string of 1's and 2's

7.1 Notations Throughout this section we fix a positive integer $m \geq 1$, and a string $\varepsilon = (l_1, \ldots, l_m) \in \{1, 2\}^m$. We make the assumption that $l_1 = 1$. The number of occurrences of 1 in the string will be denoted by n $(1 \leq n \leq m)$.

Also, for the whole section we fix a circle of radius 1 in the plane, and the points $P_1, \ldots, P_m, Q_1, \ldots, Q_m, R_1, \ldots, R_n$ on the circle, positioned as follows. First we draw $P_1, \ldots,$

P_m around the circle, equidistant and in clockwise order. Then we draw $Q_1, \ldots, Q_m$, according to the following rule: if $l_i = 1$, we put Q_i on the arc of circle going from P_i to P_{i+1}, such that the length of the arc from P_i to Q_i is $1/3$ of the length of the arc from P_i to P_{i+1} (i.e. $2\pi/3m$); and if $l_i = 2$, we put Q_i on the arc of circle from P_{i-1} to P_i, such that the length of the arc $Q_i P_i$ is $2\pi/3m$ (all arcs described clockwisely). It is convenient to view the points $Q_1, \ldots, Q_m$ as colored, namely we will say that Q_i is red whenever $l_i = 1$, and that it is blue whenever $l_i = 2$. (Note that Q_1 is red, since it is assumed that $l_1 = 1$.) Finally, we inspect the n points Q_i that are red, clockwisely and starting from Q_1, and we second-name as $R_1, \ldots, R_n$, in this order (every Q_i which is red is hence at the same time an R_j for some j).

For example, the next figure shows how our circular picture looks if $m = 8$ and $\varepsilon = (1, 2, 2, 1, 1, 2, 1, 2)$.

Figure 2.

7.2 The "complementation" maps C_Q and C_R Using the circular picture associated to the string ε, we will define two maps $C_Q : NC(m) \to NC(m)$, and $C_R : NC(m) \to NC(n)$. It is convenient to formulate the definition in terms of equivalence relations associated to partitions (recall from 2.1 that if π is a partition of $\{1, \ldots, k\}$, then the equivalence relation $\stackrel{\pi}{\sim}$ corresponding to π is simply "$i \stackrel{\pi}{\sim} j \Leftrightarrow i, j$ belong to the same block of π", $1 \leq i, j \leq k$). Given $\sigma \in NC(m)$, the equivalence relations corresponding to the partitions $C_Q(\sigma)$ of $\{1, \ldots, m\}$ and $C_R(\sigma)$ of $\{1, \ldots, n\}$ are defined as follows:

$$\left\{ \begin{array}{l} \text{- For } 1 \leq i, j \leq m \text{ we have } i \stackrel{C_Q(\sigma)}{\sim} j \text{ if and only if there are no } 1 \leq h, k \leq m \\ \text{such that } h \stackrel{\sigma}{\sim} k \text{ and such that the line segments } Q_iQ_j \text{ and } P_hP_k \text{ have} \\ \text{non-void intersection;} \\ \\ \text{- For } 1 \leq i, j \leq n \text{ we have } i \stackrel{C_R(\sigma)}{\sim} j \text{ if and only if there are no } 1 \leq h, k \leq m \\ \text{such that } h \stackrel{\sigma}{\sim} k \text{ and such that the line segments } R_iR_j \text{ and } P_hP_k \text{ have} \\ \text{non-void intersection;} \end{array} \right. \tag{7.1}$$

We leave it to the reader to verify that $\stackrel{C_Q(\sigma)}{\sim}$ and $\stackrel{C_R(\sigma)}{\sim}$ defined in (7.1) really are equivalence relations on $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$, respectively, and moreover, that $C_Q(\sigma)$ and $C_R(\sigma)$ are indeed non-crossing partitions. All these verifications reduce to the same geometric argument, that can be stated as follows: let X, Y be points on the circle, and let Z be either on the circle or in the open disk enclosed by it, such that $\{X, Y, Z\} \cap \{P_1, \ldots, P_m\} = \emptyset$; if the segment P_hP_k (for some $1 \leq h < k \leq m$) intersects XY, then it must also intersect $XZ \cup YZ$.

The relevance of the complementation maps C_Q and C_R for our discussion comes from the following

7.3 Proposition: Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and let $a_1, a_2, p_1, p_2 \in \mathcal{A}$ be such that $\{a_1, a_2\}$ is free from $\{p_1, p_2\}$ in $(\mathcal{A}, \varphi)$. Define $x_1 = a_1p_1, x_2 = p_2a_2, y_1 = a_1p_1p_2, y_2 = a_2$. Then we have:

$$\varphi(x_{l_1}x_{l_2}\cdots x_{l_m}) \ =$$

$$= \sum_{\sigma \in NC(m)} [\text{coef } (l_1, \ldots, l_m); \sigma](R(\mu_{a_1,a_2})) \cdot [\text{coef } (l_1, \ldots, l_m); C_Q(\sigma)](M(\mu_{p_1,p_2})) \tag{7.2}$$

and

$$\varphi(y_{l_1}y_{l_2}\cdots y_{l_m}) \ =$$

$$= \sum_{\sigma \in NC(m)} [\text{coef } (l_1, \ldots, l_m); \sigma](R(\mu_{a_1,a_2})) \cdot [\text{coef } (n); C_R(\sigma)](M(\mu_{p_1 \cdot p_2})) \tag{7.3}$$

(where $\varepsilon = (l_1, \ldots, l_m)$ is the string fixed in the beginning of the section; the notations for the series $M(\mu_{\ldots}), R(\mu_{\ldots})$, and for their coefficients are as in Section 3.2).

Proof We will only show the proof of (7.2), the one for (7.3) is similar. This kind of argument has been used several times before, in connection to the Kreweras complementation map K (instead of C_Q, C_R) - see 3.4 in [15], 3.4 in [9], 3.11 in [10].

If in $x_{l_1}x_{l_2}\cdots x_{l_m}$ we write back each x_1 as a_1p_1 and each x_2 as p_2a_2, we obtain a monomial of length $2m$ in a_1, a_2, p_1, p_2. By looking just at the a's in this monomial, we see that they are $a_{l_1}, a_{l_2}, \ldots, a_{l_m}$, exactly in this order, and placed on a certain set of positions $I \subseteq \{1, 2, \ldots, 2m\}$, with $|I| = m$. It is also clear that on the complementary set of positions $J = \{1, 2, \ldots, 2m\} \setminus I$ of our monomial of length $2m$ we have $p_{l_1}, p_{l_2}, \ldots, p_{l_m}$, exactly in this order. So we can say that $x_{l_1}x_{l_2}\cdots x_{l_m}$ is obtained by shuffling together $a_{l_1}a_{l_2}\cdots a_{l_m}$ and $p_{l_1}p_{l_2}\cdots p_{l_m}$, where the a's have to sit on the positions indicated by I, and the p's have to sit on the positions indicated by $J = \{1, 2, \ldots, 2m\} \setminus I$.

Now, the value of φ on the monomial of length $2m$ discussed in the preceding paragraph can be viewed as a coefficient of length $2m$ of the series $M(\mu_{a_1,a_2,p_1,p_2})$. We write this coefficient as a summation over $NC(2m)$, by using the moment-cumulant formula (3.12). Because of the assumption that $\{a_1, a_2\}$ is free from $\{p_1, p_2\}$ (which implies that we have $[R(\mu_{a_1,a_2,p_1,p_2})](z_1, z_2, z_3, z_4) = [R(\mu_{a_1,a_2})](z_1, z_2) + [R(\mu_{p_1,p_2})](z_3, z_4)$), the summation over $NC(2m)$ we have arrived to has a lot of vanishing terms; a partition $\theta \in NC(2m)$ can in fact bring a non-zero contribution to the sum if and only if each block of θ either is contained in I or is contained in J (with I, J the sets of positions of the preceding paragraph). This

brings us to the formula:

$$\varphi(x_{l_1} x_{l_2} \cdots x_{l_m}) \ =$$

$$= \sum_{\substack{\sigma,\tau \in NC(m) \\ I,J-compatible}} [\text{coef } (l_1, \ldots, l_m); \sigma](R(\mu_{a_1,a_2})) \cdot [\text{coef } (l_1, \ldots, l_m); \tau](R(\mu_{p_1,p_2})); \quad (7.4)$$

in (7.4), the fact that $\sigma, \tau \in NC(m)$ are I, J-compatible has the meaning that if we transport σ from $\{1, \ldots, m\}$ onto I and we transport τ from $\{1, \ldots, m\}$ onto $J = \{1, 2, \ldots, 2m\} \setminus I$, then the partition of $\{1, 2, \ldots, 2m\}$ which is obtained in this way is still non-crossing.

If we now look back at how the complementation map $C_Q : NC(m) \to NC(m)$ was defined, it is immediate that the I, J-compatibility of $\sigma, \tau \in NC(m)$ is equivalent to the condition $\tau \leq C_Q(\sigma)$. This means that (7.4) can be continued with:

$$\sum_{\sigma \in NC(m)} [\text{coef } (l_1, \ldots, l_m); \sigma](R(\mu_{a_1,a_2})) \cdot \{ \sum_{\substack{\tau \in NC(m), \\ \tau \leq C_Q(\sigma)}} [\text{coef } (l_1, \ldots, l_m); \tau](R(\mu_{p_1,p_2})) \}. \quad (7.5)$$

Finally, we note that

$$\sum_{\substack{\tau \in NC(m), \\ \tau \leq C_Q(\sigma)}} [\text{coef } (l_1, \ldots, l_m); \tau](R(\mu_{p_1,p_2})) \ = \ [\text{coef } (l_1, \ldots, l_m); C_Q(\sigma)](M(\mu_{p_1,p_2})), \quad (7.6)$$

as it follows by a repeated utilization of the moment-cumulant formula (3.12). Substituting (7.6) in (7.5) brings us to the right-hand side of (7.2), and concludes the proof. **QED**

7.4 Corollary Let $\varepsilon = (l_1, \ldots, l_m)$ be the string of 1's and 2's fixed in 7.1, and recall that $1 \leq n \leq m$ is the number of occurrences of 1 in the string. Let on the other hand $(\mathcal{A}, \varphi)$ be a non-commutative probability space, with φ a trace, and consider $a_1, a_2, p_1, p_2 \in \mathcal{A}$ such that (a_1, a_2) is an R-diagonal pair, and such that $\{p_1, p_2\}$ is free from $\{a_1, a_2\}$. Define $x_1 = a_1 p_1, x_2 = p_2 a_2, y_1 = a_1 p_1 p_2, y_2 = a_2$. If it is not true that m is even and $n = m/2$, then $\varphi(x_{l_1} x_{l_2} \cdots x_{l_m}) = \varphi(y_{l_1} y_{l_2} \cdots y_{l_m}) = 0$.

Proof If it is not true that m is even and $n = m/2$, then due to the particular form of $R(\mu_{a_1,a_2})$ we have that $[\text{coef } (l_1, \ldots, l_m); \sigma](R(\mu_{a_1,a_2})) = 0$ for every $\sigma \in NC(m)$. But then all the terms of the summations on the right-hand side of (7.2), (7.3) are equal to zero. **QED**

Corollary 7.4 shows that the equality (6.4), whose proof is the goal of the present section, takes place in a trivial way unless we impose the following

7.5 Supplementary hypothesis: From now on, until the end of section, we will assume that the string $\varepsilon = (l_1, \ldots, l_m)$ fixed in 7.1 contains an equal number of 1's and of 2's (i.e., m is even and $n = m/2$).

In this case, the quantities $\varphi(x_{l_1} x_{l_2} \cdots x_{l_m})$ and $\varphi(y_{l_1} y_{l_2} \cdots y_{l_m})$ mentioned in 7.4 are in general non-zero, and in order to verify their equality we will need to prove some facts concerning "ε-alternating partitions".

7.6 Definition A partition $\sigma \in NC(m)$ will be called *ε-alternating* (where $\varepsilon = (l_1, \ldots, l_m)$ is the string of 7.1) if for every block $B = \{i_1 < i_2 < \cdots < i_k\}$ of σ we have that $l_{i_1} \neq l_{i_2}, \ldots, l_{i_{k-1}} \neq l_{i_k}, l_{i_k} \neq l_{i_1}$. We denote the set of all ε-alternating partitions in $NC(m)$ by $NC_{\varepsilon-alt}(m)$.

Other two (obviously equivalent) ways of stating that $\sigma \in NC(m)$ is ε-alternating are:

- in the terminology of 2.2: we have $l_i \neq l_j$ whenever $1 \leq i < j \leq m$ belong to the same block of σ, and are consecutive in that block; or

- in the terminology of 3.2.2^o: for every block B of σ, the $|B|$-tuple $(l_1, \ldots, l_m)|B$ is of the form $(1, 2, 1, 2, \ldots, 1, 2)$ or $(2, 1, 2, 1, \ldots, 2, 1)$.

Note that from 7.6 (or any of the two equivalent reformulations stated above) it follows that every block of $\sigma \in NC_{\varepsilon-alt}(m)$ has an even number of elements; i.e., $NC_{\varepsilon-alt}(m)$ is a subset of $NC_{p-alt}(m)$ discussed in Section 4.

7.7 Proposition The complementation map $C_Q : NC(m) \to NC(m)$ sends $NC_{\varepsilon-alt}(m)$ into itself.

Proof Consider the picture with $2m$ points $P_1, \ldots, P_m, Q_1, \ldots, Q_m$ sitting on a circle of radius 1, which was used to define C_Q in 7.2. We denote by $\underline{D}$ the closed disk enclosed by the circle. During this proof we will be particularly interested in a certain type of convex subsets of $\underline{D}$, described as follows. Let $i_1, j_1, \ldots, i_k, j_k$ $(1 \leq k \leq m/2)$ be a family of $2k$ distinct indices in $\{1, \ldots, m\}$, with the property that for every $1 \leq h \leq k$, all the points $P_{i_1}, P_{j_1}, \ldots, P_{i_{h-1}}, P_{j_{h-1}}, P_{i_{h+1}}, P_{j_{h+1}}, \ldots, P_{i_k}, P_{j_k}$ lie in the same open half-plane $\underline{S}_h$

determined by the line through P_{i_h} and P_{j_h}. Then the set $\underline{X} \stackrel{def}{=} \underline{D} \cap \underline{S}_1 \cap \cdots \cap \underline{S}_k$ will be called the *trimmed disk* determined by $i_1, j_1, \ldots, i_k, j_k$. A picture of a trimmed disk (exemplified for $m = 12$) is shown in Figure 3:

Figure 3: An example of trimmed disk, when $m = 12$.

The trimmed disk $\underline{X}$ determined by $i_1, j_1, \ldots, i_k, j_k$ is a convex set (neither open nor closed). The points $P_{i_1}, P_{j_1}, \ldots, P_{i_k}, P_{j_k}$ will be called the *vertices* of $\underline{X}$ (and can be identified as those P_i, $1 \leq i \leq m$, which lie in the closure of $\underline{X}$ but not in $\underline{X}$ itself). The boundary of $\underline{X}$ consists of $2k$ "edges"; k of these edges are the line segments $P_{i_h}P_{j_h}$, $1 \leq h \leq k$, and the other k edges are arcs of the circle. Note that when we travel around the boundary of $\underline{X}$, the rectilinear and curvilinear edges alternate.

Let us now fix for the rest of the proof a partition $\sigma \in NC_{\varepsilon-alt}(m)$, about which we want to show that $C_Q(\sigma)$ is also in $NC_{\varepsilon-alt}(m)$.

For every block B of σ we denote by $\underline{H}_B$ the closed convex hull of the points $\{P_i \mid i \in B\}$; $\underline{H}_B$ is thus a closed convex polygon with $|B|$ vertices, inscribed in the circle. The polygons $(\underline{H}_B \ ; \ B$ block of $\sigma)$ are disjoint, due to the fact that σ is non-crossing. We look at the connected components of the complement $\underline{D} \setminus \cup_B \underline{H}_B$. Each of these connected components is a trimmed disk; the proof of this fact is most easily done by taking the polygons $\underline{H}_B$ out of $\underline{D}$ one by one, and using an induction argument.

Given $1 \leq i, j \leq m$, we have that i and j are in the same block of $C_Q(\sigma)$ if and only if the points Q_i and Q_j lie in the same connected component of $\underline{D} \setminus \cup_B \underline{H}_B$; this follows immediately from the definition of the map C_Q in (7.1), where at "$\Leftarrow$" we also use the fact that the connected components of $\underline{D} \setminus \cup_B \underline{H}_B$ are convex. The blocks of $C_Q(\sigma)$ are

thus found by looking at the connected components $\underline{X}$ of $\underline{D} \setminus \cup_B \underline{H}_B$, with the property that $\underline{X} \cap \{Q_1, \ldots, Q_m\} \neq \emptyset$. Hence in order to show that $C_Q(\sigma)$ is ε-alternating, we have to consider such a connected component $\underline{X}$, and prove that when we travel around the boundary of $\underline{X}$ we meet an even number of points Q_i, which are of alternating colors. (Recall from 7.1 that the points $Q_1, \ldots, Q_m$ are colored - Q_i is red when $l_i = 1$, and is blue when $l_i = 2$.)

We fix for the rest of the proof a connected component $\underline{X}$ of $\underline{D} \setminus \cup_B \underline{H}_B$, such that $\underline{X} \cap \{Q_1, \ldots, Q_m\} \neq \emptyset$. As remarked earlier, $\underline{X}$ is a trimmed disk. Note that all the curvilinear edges of $\underline{X}$ have length $2\pi/m$ (because otherwise $\underline{X}$ would also contain some of the points $P_i,\ 1 \leq i \leq m$; but all the points P_i are in $\cup_B \underline{H}_B$). The points Q_i that are to be found in X are all lying on the curvilinear edges of the boundary of $\underline{X}$. On the other hand, if the arc from P_i to P_{i+1} is such a curvilinear edge, then it contains zero, one or two points from $\{Q_1, \ldots, Q_m\}$, according to what are l_i and l_{i+1} :

- if $l_i = l_{i+1} = 1$, then it contains Q_i;
- if $l_i = l_{i+1} = 2$, then it contains Q_{i+1};
- if $l_i = 1, l_{i+1} = 2$, then it contains Q_i and Q_{i+1};
- if $l_i = 2, l_{i+1} = 1$, then it contains no point from $\{Q_1, \ldots, Q_m\}$.

The conclusion of the preceding paragraph is that if we want to find the points from $\{Q_1, \ldots, Q_m\}$ that lie in $\underline{X}$, what we have to do is inspect the vertices of $\underline{X}$, and see for each such vertex P_i whether the corresponding Q_i is on a curvilinear edge of $\underline{X}$ or not.

But if we are to inspect the vertices of $\underline{X}$, the way we want to do this is by pairing them along the rectilinear edges of $\underline{X}$. (Recall that the rectilinear and curvilinear edges of $\underline{X}$ are alternating, so that the rectilinear edges contain all the vertices of $\underline{X}$, without repetitions.) So let us go around the boundary of $\underline{X}$, clockwisely, and take one by one the rectilinear edges which occur. Always for such an edge, call it $P_i P_j$ (where P_i comes first in clockwise order), we have that $l_i \neq l_j$ - here is the place where we are using the hypothesis $\sigma \in NC_{\varepsilon-alt}(m)$. It is easily seen that:

- if $l_i = 1, l_j = 2$, then P_i and P_j do not produce points Q_i, Q_j in $\underline{X}$;
- if $l_i = 2, l_j = 1$, then both Q_i and Q_j are in $\underline{X}$; moreover, when recording clockwisely what is $\{Q_1, \ldots, Q_m\} \cap \underline{X}$, these two points Q_i, Q_j will be consecutive, of different colors, and the blue one will come first.

Hence we can organize the points in $\{Q_1, \ldots, Q_m\} \cap \underline{X}$ in pairs, such that when we travel clockwisely around the boundary of $\underline{X}$ the points from each pair are consecutive, of different

colors, and with the blue one coming first. But this clearly implies that $\{Q_1, \ldots, Q_m\} \cap \underline{X}$ has an even number of points, and of alternating colors - as it was to be shown. **QED**

7.8 Corollary If $\sigma \in NC_{\varepsilon-alt}(m)$, then the partitions $C_Q(\sigma) \in NC(m)$ and $C_R(\sigma) \in NC(n)$ defined in 7.2 have the same number of blocks, say k; moreover, we can write them as $C_Q(\sigma) = \{B_1, \ldots, B_k\}$ and $C_R(\sigma) = \{A_1, \ldots, A_k\}$, in such a way that $|B_j| = 2|A_j|$ for every $1 \leq j \leq k$ (recall that $m = 2n$, due to the supplementary hypothesis made in 7.5).

Proof Consider the closed convex polygons $(\underline{H}_B \ ; \ B$ block of $\sigma)$ defined as in the proof of Proposition 7.7. As pointed out in the named proof, the blocks of $C_Q(\sigma)$ are in one-to-one correspondence with the connected components $\underline{X}$ of $\underline{D} \setminus \cup_B \underline{H}_B$, having the property that $\underline{X} \cap \{Q_1, \ldots, Q_m\} \neq \emptyset$. In exactly the same way it is seen that the blocks of $C_R(\sigma)$ are in one-to-one correspondence with the connected components $\underline{X}$ of $\underline{D} \setminus \cup_B \underline{H}_B$, having the property that $\underline{X} \cap \{R_1, \ldots, R_n\} \neq \emptyset$. So the proof will be over if we can show that for an arbitrary connected component $\underline{X}$ of $\underline{D} \setminus \cup_B \underline{H}_B$ we have:

$$\begin{cases} \text{(a) } \underline{X} \cap \{Q_1, \ldots, Q_m\} \neq \emptyset \Leftrightarrow \underline{X} \cap \{R_1, \ldots, R_n\} \neq \emptyset; \\ \\ \text{(b) if the equivalent statements of (a) are holding,} \\ \text{then } |\underline{X} \cap \{Q_1, \ldots, Q_m\}| = 2|\underline{X} \cap \{R_1, \ldots, R_n\}|. \end{cases} \tag{7.7}$$

But recall now that the set $\{R_1, \ldots, R_n\}$ is nothing else than the set of those points from $\{Q_1, \ldots, Q_m\}$ that are colored in red. Hence (7.7) can be restated as:

$$\begin{cases} \text{(a) there are some points } Q_i, \ 1 \leq i \leq m \text{ in } \underline{X} \text{ if and only if} \\ \text{there are some red points } Q_i, \ 1 \leq i \leq m \text{ in } \underline{X}; \\ \\ \text{(b) if the equivalent statements of (a) are holding, then the total} \\ \text{number of points } Q_i \text{ in } \underline{X} \text{ is twice the number of red points } Q_i \text{ in } \underline{X}. \end{cases} \tag{7.8}$$

But (7.8) is a clear consequence of the fact that $C_Q(\sigma) \in NC_{\varepsilon-alt}(m)$, proved in 7.7. **QED**

The next proposition concludes the proof of (6.4) (and hence of Proposition 6.3 and of Theorem 1.5).

7.9 Proposition Let $\varepsilon = (l_1, \ldots, l_m)$ be the string of 1's and 2's fixed in 7.1; recall that according to 7.5, m is even, and the number of occurrences of both 1 and 2 in the string is $n = m/2$. Let on the other hand $(\mathcal{A}, \varphi)$ be a non-commutative probability space, with φ a trace, and consider $a_1, a_2, p_1, p_2 \in \mathcal{A}$ such that (a_1, a_2) is an R-diagonal pair, and such

that $\{p_1, p_2\}$ is free from $\{a_1, a_2\}$. Define $x_1 = a_1p_1, x_2 = p_2a_2, y_1 = a_1p_1p_2, y_2 = a_2$. Then $\varphi(x_{l_1}x_{l_2}\cdots x_{l_m}) = \varphi(y_{l_1}y_{l_2}\cdots y_{l_m})$.

Proof We consider the expressions of $\varphi(x_{l_1}x_{l_2}\cdots x_{l_m})$ and $\varphi(y_{l_1}y_{l_2}\cdots y_{l_m})$ via summations over $NC(m)$, as shown in Eqns.(7.2),(7.3); we will prove that for every $\sigma \in NC(m)$, the terms indexed by σ in the two sums of (7.2) and (7.3) coincide. If $\sigma \notin NC_{\varepsilon-alt}(m)$ this is clear, because $[\text{coef }(l_1,\ldots,l_m);\sigma](R(\mu_{a_1,a_2})) = 0$ (due to the particular form of $R(\mu_{a_1,a_2})$), hence the terms indexed by σ in (7.2) and (7.3) are both equal to zero. For $\sigma \in NC_{\varepsilon-alt}(m)$, it suffices to show that

$$[\text{coef }(l_1,\ldots,l_m);C_Q(\sigma)](M(\mu_{p_1,p_2})) \;=\; [\text{coef }(m/2);C_R(\sigma)](M(\mu_{p_1\cdot p_2})). \tag{7.9}$$

According to Corollary 7.8, we can write $C_Q(\sigma) = \{B_1,\ldots,B_k\}$, $C_R(\sigma) = \{A_1,\ldots,A_k\}$, in such a way that $|B_j| = 2|A_j|$ for every $1 \leq j \leq k$. On the other hand, due to the fact that $C_Q(\sigma)$ is also ε-alternating (by Proposition 7.7), we see that

$$[\text{coef }(l_1,\ldots,l_m);C_Q(\sigma)](M(\mu_{p_1,p_2})) \;=\; \varphi((p_1p_2)^{|B_1|/2})\cdots\varphi((p_1p_2)^{|B_k|/2}) \tag{7.10}$$

(in (7.10) the fact that φ is a trace is also used). But it is clear that

$$[\text{coef }(m/2);C_R(\sigma)](M(\mu_{p_1\cdot p_2})) \;=\; \varphi((p_1p_2)^{|A_1|})\cdots\varphi((p_1p_2)^{|A_k|/2}); \tag{7.11}$$

the right-hand sides of (7.10) and (7.11) coincide, which establishes (7.9). **QED**

8. The proof of Theorem 1.13

8.1 Notations In this section $(\mathcal{A},\varphi)$ is a fixed non-commutative probability space, such that φ is a trace, and $u, p_{1,1}, p_{1,2},\ldots,p_{k,1},p_{k,2} \in \mathcal{A}$ are elements satisfying the conditions (i), (ii), (iii) of the Theorem 1.13. It will be handier to prove that the sets $\{p_{1,1}u^{-1}, up_{1,2}\},\ldots,\{p_{k,1}u^{-1}, up_{k,2}\}$ are free (this is equivalent to the statement of Theorem 1.13, by swapping $p_{j,1}$ with $p_{j,2}$ for every $1 \leq j \leq k$).

We make the notations $p_{j,1}u^{-1} \overset{def}{=} a_{j,1}$, $up_{j,2} \overset{def}{=} a_{j,2}$, $1 \leq j \leq k$. In order to start the proof that the sets $\{a_{1,1}, a_{1,2}\},\ldots,\{a_{k,1}, a_{k,2}\}$ are free in $(\mathcal{A},\varphi)$, we will use an abstract nonsense construction: we consider (and fix for the whole section) another non-commutative probability space $(\mathcal{B},\psi)$, with ψ a trace, and elements $b_{1,1}, b_{1,2},\ldots,b_{k,1},b_{k,2} \in \mathcal{B}$ such that:

(j) $\mu_{b_{j,1},b_{j,2}} = \mu_{a_{j,1},a_{j,2}}$, for every $1 \leq j \leq k$ (where the joint distributions $\mu_{b_{j,1},b_{j,2}}$ and $\mu_{a_{j,1},a_{j,2}}$ are considered in $(\mathcal{B}, \psi)$ and $(\mathcal{A}, \varphi)$, respectively).

(jj) the sets $\{b_{1,1}, b_{1,2}\}, \ldots, \{b_{k,1}, b_{k,2}\}$ are free in $(\mathcal{B}, \psi)$.

Such a construction can of course be done, for instance we can take $(\mathcal{B}, \psi)$ to be the free product $\star_{j=1}^{k}(\mathcal{A}_j, \varphi|\mathcal{A}_j)$, where $\mathcal{A}_j$ is the unital algebra generated by $\{a_{j,1}, a_{j,2}\}$ in $\mathcal{A}$, and then we can take $b_{j,1}, b_{j,2}$ to be just $a_{j,1}, a_{j,2}$, but viewed in $\mathcal{B}$. (Note that $\star_{j=1}^{k}(\varphi|\mathcal{A}_j)$ is a trace, because each $\varphi|\mathcal{A}_j$ is so, and by Proposition 2.5.3 of [21] - this justifies why we could assume that ψ is a trace.)

In this setting, our goal is to show that

$$\mu_{a_{1,1},a_{1,2},\ldots,a_{k,1},a_{k,2}} \;=\; \mu_{b_{1,1},b_{1,2},\ldots,b_{k,1},b_{k,2}}. \tag{8.1}$$

Indeed, as it is clear from its very definition, the freeness of a family of subsets can be read from the joint distribution of the union of those subsets; so in the presence of (8.1), the freeness of $\{a_{1,1}, a_{1,2}\}, \ldots, \{a_{k,1}, a_{k,2}\}$ is implied by the one of $\{b_{1,1}, b_{1,2}\}, \ldots, \{b_{k,1}, b_{k,2}\}$.

8.2 Remark For every $1 \leq j \leq k$, the pair $(a_{j,1}, a_{j,2})$ - and hence $(b_{j,1}, b_{j,2})$ too - is R-diagonal with determining series

$$f_j \;\stackrel{def}{=}\; R(\mu_{p_{j,1}\cdot p_{j,2}}) \boxed{\star}\, Moeb. \tag{8.2}$$

This follows from Theorem 1.5 and Proposition 1.7 (and the fact that $a_{j,1} \cdot a_{j,2} = p_{j,1} \cdot p_{j,2}$).

We have, in other words, that

$$[R(\mu_{a_{j,1},a_{j,2}})](z_{j,1}, z_{j,2}) \;=\; [R(\mu_{b_{j,1},b_{j,2}})](z_{j,1}, z_{j,2}) \;=\; f_j(z_{j,1} \cdot z_{j,2}) + f_j(z_{j,2} \cdot z_{j,1}), \tag{8.3}$$

for every $1 \leq j \leq k$. The freeness of $\{b_{1,1}, b_{1,2}\}, \ldots, \{b_{k,1}, b_{k,2}\}$ enables us to obtain from (8.3) the formula for the R-transform $R(\mu_{b_{1,1},b_{1,2},\ldots,b_{k,1},b_{k,2}})$, this is the series of $2k$ variables

$$f(z_{1,1}, z_{1,2}, \ldots, z_{k,1}, z_{k,2}) \;\stackrel{def}{=}\; \sum_{j=1}^{k} f_j(z_{j,1} \cdot z_{j,2}) + f_j(z_{j,2} \cdot z_{j,1}). \tag{8.4}$$

(Of course, we don't have the analogue of this fact for the a's - if we would, the proof would be finished.)

A way of re-interpreting (8.2) which will be used later on is the following:

8.3 Lemma For every $1 \leq j \leq k$ and every $n \geq 1$, the coefficients of $(z_{j,1} \cdot z_{j,2})^n$, $(z_{j,2} \cdot z_{j,1})^n$ in $[R(\mu_{b_{j,1},b_{j,2}})](z_{j,1}, z_{j,2})$ and $[R(\mu_{p_{j,1},p_{j,2}})](z_{j,1}, z_{j,2})$ are (all four of them) equal to the coefficient of order n in the series f_j of (8.2).

Proof For $R(\mu_{b_{j,1},b_{j,2}})$ this has been explicitly written in (8.3), while for $R(\mu_{p_{j,1},p_{j,2}})$ we use Proposition 5.3, the hypothesis that the pair $(p_{j,1}, p_{j,2})$ is diagonally balanced, and the form of f_j in (8.2). **QED**

8.4 The approach to the proof of Theorem 1.13 will follow from now on the same pattern as the one used for the proof of Theorem 1.5. Indeed, Eqn.(8.1) simply says that for every $m \geq 1$ and $l_1, \ldots, l_m \in \{1, 2\}$, $h_1, \ldots, h_m \in \{1, \ldots, k\}$, we have

$$\varphi(a_{h_1,l_1} \cdots a_{h_m,l_m}) \; = \; \psi(b_{h_1,l_1} \cdots b_{h_m,l_m}). \tag{8.5}$$

In the rest of the section we will work on proving (8.5), in a way which parallels the development of Section 7. We will first obtain formulas expressing the two sides of (8.5) as summations over $NC(m)$. From these formulas it will be clear that both sides of (8.5) are equal to zero, unless $(l_1, \ldots, l_m) \in \{1, 2\}^m$ satisfies a certain balancing condition - in fact the same as the one stated in 7.5. Starting from that point, we will assume that the balancing condition holds, and in order to complete the proof we will need to throw in a "geometrical" argument concerning the circular picture of a non-crossing partition.

Both sides of (8.5) are invariant under cyclic permutations of the monomials involved (because φ and ψ are traces). Thus if we assume in (8.5) that $l_1 = 1$, we are in fact only missing the case when $l_1 = l_2 = \cdots = l_m = 2$. But in the latter case we have:

8.5 Lemma $\varphi(a_{h_1,2} a_{h_2,2} \cdots a_{h_m,2}) = \psi(b_{h_1,2} b_{h_2,2} \cdots b_{h_m,2}) = 0.$

Proof For the a's: $\varphi(a_{h_1,2} a_{h_2,2} \cdots a_{h_m,2}) = \varphi(up_{h_1,2} up_{h_2,2} \cdots up_{h_m,2})$ can be written as a coefficient of the series $M(\mu_{up_{h_1,2}, up_{h_2,2}, \ldots, up_{h_m,2}})$. But this series is identically zero; indeed, by using Eqn.(3.16) and the fact that u is free from $\{p_{1,2}, \ldots, p_{k,2}\}$), we get:

$$M(\mu_{up_{h_1,2}, up_{h_2,2}, \ldots, up_{h_m,2}}) \; = \; M(\mu_{\underbrace{u, \ldots, u}_{m}}) \boxed{\star} R(\mu_{p_{h_1,2}, p_{h_2,2}, \ldots, p_{h_m,2}}).$$

It is obvious, however, that $M(\mu_{u,\ldots,u}) = 0$, and that in general we have $0 \boxed{\star} f = 0$.

For the b's: $b_{h,2}$ is a part of the R-diagonal pair $(b_{h,1}, b_{h,2})$, and therefore, as remarked in the final paragraph of Section 6, we must have $\psi(b^n_{h,2}) = 0$ for every $1 \leq h \leq k$ and $n \geq 1$. But then, if we also take into account that $b_{1,2}, b_{2,2}, \ldots, b_{k,2}$ are free, the equality $\psi(b_{h_1,2} b_{h_2,2} \cdots b_{h_m,2}) = 0$ follows directly from the definition of freeness in (1.1). **QED**

8.6 Notations From now on, and until the end of the section, we fix: $m \geq 1$; $l_1, \ldots, l_m \in \{1,2\}^m$ such that $l_1 = 1$; and $h_1, \ldots, h_m \in \{1, \ldots, k\}$. Our goal is to prove (8.5) for this fixed set of data.

We denote the m-tuple $(l_1, \ldots, l_m) \in \{1,2\}^m$ by ε, and we denote by n $(1 \leq n \leq m)$ the number of occurrences of 1 in ε.

We will use the geometrical objects constructed in Sections 7.1, 7.2 above. That is, we consider again the circle of radius 1 and the points $P_1, \ldots, P_m, Q_1, \ldots, Q_m$ sitting on it, and positioned in the way described in 7.1; and we consider again the complementation map $C_Q : NC(m) \to NC(m)$ described in 7.2.

8.7 Remark Consider the product $a_{h_1,l_1} a_{h_2,l_2} \cdots a_{h_m,l_m}$ appearing in the right-hand side of (8.5). If in this product we write back each $a_{h_i,1}$ as $p_{h_i,1} u^{-1}$ and each $a_{h_i,2}$ as $u p_{h_i,2}$, we obtain an expression (monomial) of length $2m$ in $p_{1,1}, p_{1,2}, \ldots, p_{k,1}, p_{k,2}, u, u^{-1}$. By looking just at the p's in the latter monomial, we see that they are $p_{h_1,l_1}, p_{h_2,l_2}, \ldots, p_{h_m,l_m}$, exactly in this order, and placed on a certain set of positions $I \subseteq \{1, 2, \ldots, 2m\}$, with $|I| = m$. On the complementary set of positions $J = \{1, 2, \ldots, 2m\} \setminus I$ of our monomial of length $2m$ we have factors of u and u^{-1}; we denote them as $u^{\lambda_1}, u^{\lambda_2}, \ldots, u^{\lambda_m}$, in the order in which they appear from left to right. It is clear that λ_i, $1 \leq i \leq m$, is entirely determined by l_i, via the formula: $l_i = 1 \Rightarrow \lambda_i = -1$, $l_i = 2 \Rightarrow \lambda_i = +1$.

We can thus say that the product $a_{h_1,l_1} a_{h_2,l_2} \cdots a_{h_m,l_m}$ is obtained by shuffling together $p_{h_1,l_1} p_{h_2,l_2} \cdots p_{h_m,l_m}$ and $u^{\lambda_1} u^{\lambda_2} \cdots u^{\lambda_m}$, where the p's have to sit on the positions indicated by I, and the u's have to sit on the positions indicated by $J = \{1, 2, \ldots, 2m\} \setminus I$. Note that I and J are exactly the same as the ones appearing in the proof of Proposition 7.3.

8.8 Proposition In the notations established in 8.1, 8.6, 8.7, we have:

$$\varphi(a_{h_1,l_1} a_{h_2,l_2} \cdots a_{h_m,l_m}) =$$

$$= \sum_{\sigma \in NC(m)} [\text{coef } ((h_1, l_1), \ldots, (h_m, l_m)); \sigma](R(\mu_{p_{1,1}, p_{1,2}, \ldots, p_{k,1}, p_{k,2}})) \cdot \tag{8.6}$$

$$[\text{coef } (\lambda_1, \ldots, \lambda_m); C_Q(\sigma)](M(\mu_{u,u^{-1}}))$$

and

$$\psi(b_{h_1,l_1}b_{h_2,l_2}\cdots b_{h_m,l_m}) \ =$$

$$= \sum_{\sigma\in NC(m)} [\mathrm{coef}\ ((h_1,l_1),\ldots,(h_m,l_m));\sigma](R(\mu_{b_{1,1},b_{1,2},\ldots,b_{k,1},b_{k,2}})). \tag{8.7}$$

(In (8.6), (8.7) the series $R(\mu_{p_{1,1},p_{1,2},\ldots,p_{k,1},p_{k,2}})$ and $R(\mu_{b_{1,1},b_{1,2},\ldots,b_{k,1},b_{k,2}})$ are acting in the variables $z_{1,1},z_{1,2},\ldots,z_{k,1},z_{k,2}$ - same as in (8.4), for instance. The series $M(\mu_{u,u^{-1}})$ is viewed as acting in the two variables z_{+1} and z_{-1}.)

Proof Equation (8.7) is just the moment-cumulant formula, see Section 3.5 above.

The argument proving (8.6) is very similar to the one used in the proof of Proposition 7.3, and for this reason we will only outline its main steps. We view $\varphi(a_{h_1,l_1}a_{h_2,l_2}\cdots a_{h_m,l_m})$ as a coefficient of length $2m$ of the series $M(\mu_{p_{1,1},p_{1,2},\ldots,p_{k,1},p_{k,2},u,u^{-1}})$, and we then expand it as a summation over $NC(2m)$ by using the moment-cumulant formula (i.e., the appropriate version of Eqn.(3.12) in 3.5). Due to the fact that $\{p_{1,1},p_{1,2},\ldots,p_{k,1},p_{k,2}\}$ is free from $\{u,u^{-1}\}$, what we arrive to is the formula (paralleling (7.4) in the proof of 7.3):

$$\varphi(a_{h_1,l_1}a_{h_2,l_2}\cdots a_{h_m,l_m}) \ =$$

$$= \sum_{\substack{\sigma,\tau\in NC(m)\\ I,J-compatible}} [\mathrm{coef}\ ((h_1,l_1),\ldots,(h_m,l_m));\sigma](R(\mu_{p_{1,1},p_{1,2},\ldots,p_{k,1},p_{k,2}}))\cdot \tag{8.8}$$

$$[\mathrm{coef}\ (\lambda_1,\ldots,\lambda_m);\tau)](R(\mu_{u,u^{-1}})).$$

Then we process the right-hand side of (8.8) exactly in the way the right-hand side of (7.4) was processed in the proof of Proposition 7.3, and this leads to the right-hand side of (8.6). **QED**

8.9 Corollary Let $\varepsilon = (l_1,\ldots,l_m)$ be the string of 1's and 2's of Notations 8.6. If the number n of occurrences of 1 in ε does not equal the number $m-n$ of occurrences of 2 in ε, then $\varphi(a_{h_1,l_1}a_{h_2,l_2}\cdots a_{h_m,l_m}) = \psi(b_{h_1,l_1}b_{h_2,l_2}\cdots b_{h_m,l_m}) = 0$.

Proof For the a's: Due to how $\lambda_1,\ldots,\lambda_m$ are determined by $l_1,\ldots,l_m$ (see Remark 8.7), we have in this case that the number n of (-1)'s in $(\lambda_1,\ldots,\lambda_m)$ is different from the number $m-n$ of 1's in $(\lambda_1,\ldots,\lambda_m)$. But then it is immediate that $[\mathrm{coef}\ (\lambda_1,\ldots,\lambda_m);\tau](M(\mu_{u,u^{-1}}))$ $=0$ for every $\tau\in NC(m)$, and hence all the terms in the sum of (8.6) are vanishing.

For the b's: in (8.4) we have an explicit formula for $R(\mu_{b_{1,1},b_{1,2},\ldots,b_{k,1},b_{k,2}})$, the inspection of which makes clear (in view of the hypothesis of the present corollary) that all the terms of the sum in (8.7) are vanishing. **QED**

We therefore make, exactly as we did in 7.5, the following

8.10 Supplementary hypothesis: From now on we will assume that the string $\varepsilon = (l_1, \ldots, l_m)$ of 8.6 contains an equal number of 1's and 2's (i.e., m is even and $n = m/2$).

Moreover, we will consider the notion of an ε-*alternating* partition in $NC(m)$, which is exactly the one defined in 7.6. Recall that the set of all the ε-alternating partitions in $NC(m)$ is denoted by $NC_{\varepsilon-alt}(m)$.

8.11 Proposition A partition $\sigma \in NC(m)$ is ε-alternating if and only if it has the following properties:

1^o every block of σ has at least two elements;

2^o there exists no block B of σ such that: $|B|$ is odd, and $(l_1, \ldots, l_m)|B$ is a cyclic permutation of $(\underbrace{1, 2, \ldots, 1, 2, 1}_{|B|})$ or of $(\underbrace{2, 1, \ldots, 2, 1, 2}_{|B|})$ (where the notation $(l_1, \ldots, l_m)|B$ is in the sense of $3.2.2^o$);

3^o every block of the complementary partition $C_Q(\sigma)$ has an even number of elements.

Proof "$\Rightarrow$" As remarked immediately after the Definition 7.6, every block of σ has an even number of elements (this implies 1^o and 2^o). The same holds for $C_Q(\sigma)$, because $C_Q(\sigma)$ is also in $NC_{\varepsilon-alt}(m)$, by Proposition 7.7.

"$\Leftarrow$" By contradiction, we assume that σ is not ε-alternating. This means that we can find $1 \leq i < j \leq m$ such that i and j are in the same block of σ, and consecutive in that block (in the sense of 2.2), and such that moreover $l_i = l_j$. From all the pairs (i, j) with these properties, we choose one, (i_o, j_o), such that the length of the segment $P_{i_o}P_{j_o}$ is minimal. (We are using in this proof the circular picture involving the points $P_1, \ldots, P_m, Q_1, \ldots, Q_m$ introduced in 7.1, 7.2.)

We denote by $\underline{S}$ the open half-plane that is determined by the line through P_{i_o} and P_{j_o}, and that does not contain the center of the circle. (If the line $P_{i_o}P_{j_o}$ goes through the center of the circle, any of the two open half-planes determined by it can be chosen as $\underline{S}$.)

The minimality assumption on (i_o, j_o) implies that:

$$\left\{ \begin{array}{l} \text{if } 1 \leq i < j \leq m \text{ are in the same block of } \sigma\text{, and consecutive in that block,} \\ \text{and if in addition we have that } P_i, P_j \in \underline{S}\text{, then necessarily } l_i \neq l_j \end{array} \right. \tag{8.9}$$

(this is simply because the length of P_iP_j is strictly smaller than the one of $P_{i_o}P_{j_o}$). Note that this argument gives in fact that $l_i \neq l_j$ even if we only assume that P_i, P_j lie in the closure of $\underline{S}$, but $(i,j) \neq (i_o, j_o)$.

Let B be the block of σ containing i_o and j_o. We claim that $\{P_j \mid j \in B\} \cap \underline{S} = \emptyset$. Indeed, this is obvious if B is reduced to $\{i_o, j_o\}$, so let us assume that $|B| \geq 3$. Since i_o, j_o are consecutive in B, we have that $\{P_j \mid j \in B\}$ is contained in one of the closed half-planes determined by $P_{i_o}P_{j_o}$; so if we would assume $\{P_j \mid j \in B\} \cap \underline{S} \neq \emptyset$, then it would follow that $\{P_j \mid j \in B,\ j \neq i_o, j_o\} \subseteq \underline{S}$. But then the remark concluding the preceding paragraph would imply that $l_i \neq l_j$ whenever $i < j$ are consecutive in B, with the exception of the case when $(i,j) = (i_o, j_o)$, and this would violate property 2^o of σ.

Now, the set $\{j \mid 1 \leq j \leq m,\ P_j \in \underline{S}\}$ is either void or a union of blocks of σ, because of the non-crossing character of σ (and because of what was proved about the block $B \ni i_o, j_o$). Remark that for every block B' of σ which is contained in $\{j \mid 1 \leq j \leq m,\ P_j \in \underline{S}\}$, we must have that $|B'|$ is even; indeed, from (8.9) it follows that $l_i \neq l_j$ for every $i, j \in B'$ which are consecutive in B', and this couldn't happen if $|B'|$ would be odd. Consequently, the set $\{j \mid 1 \leq j \leq m,\ P_j \in \underline{S}\}$ has an even cardinality (possibly zero).

By recalling how the points $Q_1, \ldots, Q_m$ were constructed, we next note that $\{j \mid 1 \leq j \leq m,\ P_j \in \underline{S}\} \subset \{j \mid 1 \leq j \leq m,\ Q_j \in \underline{S}\}$, and moreover that the set-difference $\{j \mid 1 \leq j \leq m,\ Q_j \in \underline{S},\ P_j \notin \underline{S}\}$ has exactly one element, which is either i_o or j_o. The latter assertion amounts to the following two facts, both obvious: (a) if $1 \leq j \leq m$ is such that P_j does not belong to the closure of $\underline{S}$, then $Q_j \notin \underline{S}$; and (b) Q_{i_o} and Q_{j_o} lie in opposite open half-planes determined by the line $P_{i_o}P_{j_o}$, hence exactly one of them is in $\underline{S}$ (the assumption $l_{i_o} = l_{j_o}$ is of course crucial for (b)). The conclusion of this paragraph is that the cardinality $|\{j \mid 1 \leq j \leq m,\ Q_j \in \underline{S}\}| = 1 + |\{j \mid 1 \leq j \leq m,\ P_j \in \underline{S}\}|$ is an odd number.

But because of how the definition of the complementation map C_Q was made in 7.2, the set $\{j \mid 1 \leq j \leq m,\ Q_j \in \underline{S}\}$ is a union of blocks of $C_Q(\sigma)$. Hence, in view of the property 3^o of σ, $|\{j \mid 1 \leq j \leq m,\ Q_j \in \underline{S}\}|$ is an even number - contradiction. **QED**

8.12 The conclusion of the proof Recall that what we need to prove is the equality (8.5) of 8.4, for the $m \geq 1$, $l_1, \ldots, l_m \in \{1, 2\}$, $h_1, \ldots, h_m \in \{1, \ldots, k\}$ that were fixed in

8.6, and where $\varepsilon = (l_1, \ldots, l_m)$ satisfies the additional hypothesis 8.10.

Let us express both quantities $\varphi(a_{h_1,l_1} \cdots a_{h_m,l_m})$ and $\psi(b_{h_1,l_1} \cdots b_{h_m,l_m})$ appearing in (8.5) as summations over $NC(m)$, in the way shown in Proposition 8.8 (Eqns.(8.6) and (8.7), respectively). We will prove that for every $\sigma \in NC(m)$, the terms indexed by σ in the two sums of (8.6) and (8.7) coincide - this will of course imply that the sums are equal.

Up to now, the m-tuple $(h_1, \ldots, h_m) \in \{1, \ldots, k\}^m$, which is part of our data, didn't play any role. Let us view this m-tuple as a function from $\{1, \ldots, m\}$ to $\{1, \ldots, k\}$, and denote by $L_1, \ldots, L_k$ its level sets; i.e., $L_h \overset{def}{=} \{i \mid 1 \leq i \leq m,\ h_i = h\}$, for $1 \leq h \leq k$. We will say about a partition $\sigma \in NC(m)$ that it is *h-acceptable* if every block of σ is contained in one of the level sets $L_1, \ldots, L_k$; and we will denote by $NC_{h-acc}(m)$ the set of all h-acceptable partitions in $NC(m)$.

The general term of the sum in (8.7) is a product of coefficients of $R(\mu_{b_{1,1},b_{1,2},\ldots,b_{k,1},b_{k,2}})$, made in the way dictated by $\sigma \in NC(m)$ (according to the recipe of $3.2.3^o$). By using the freeness of the sets $\{b_{1,1}, b_{1,2}\}, \ldots, \{b_{k,1}, b_{k,2}\}$ and Theorem 3.6, it is easily seen that this product is zero whenever σ is not h-acceptable; while for $\sigma \in NC_{h-acc}(m)$ we get the formula:

$$[\mathrm{coef}\ ((h_1, l_1), \ldots, (h_m, l_m)); \sigma](R(\mu_{b_{1,1},b_{1,2},\ldots,b_{k,1},b_{k,2}})) = \tag{8.10}$$

$$= \prod_{\substack{1 \leq h \leq k \\ such\ that \\ L_h \neq \emptyset}} \Big\{ \prod_{\substack{B\ block\ of\ \sigma \\ such\ that \\ B \subseteq L_h}} [\mathrm{coef}\ ((h_1, l_1), \ldots, (h_m, l_m))|B](R(\mu_{b_{h,1},b_{h,2}}))\ \Big\},$$

(where the notation for the restricted $|B|$-tuple $((h_1, l_1), \ldots, (h_m, l_m))|B$ is taken from $3.2.2^o$). We next recall from 8.2 that the pair $(b_{h,1}, b_{h,2})$ is R-diagonal, for every $1 \leq h \leq k$; hence even if we assume that $\sigma \in NC_{h-acc}(m)$, the particular form of $R(\mu_{b_{h,1},b_{h,2}})$ will force the product in (8.10) to still be equal to zero, unless we also assume that the $|B|$-tuple $(l_1, \ldots, l_m)|B$ is $(1, 2, 1, 2, \ldots, 1, 2)$ or $(2, 1, 2, 1, \ldots, 2, 1)$ for every block B of σ. In other words, if we want the contribution of $\sigma \in NC(m)$ to the sum (8.7) to be non-zero, then we must also impose (besides the condition $\sigma \in NC_{h-acc}(m)$) that σ is ε-alternating.

To conclude the discussion concerning (8.7), let us consider $\sigma \in NC_{h-acc}(m) \cap NC_{\varepsilon-alt}(m)$. Then the product in (8.10) can be prelucrated by using Lemma 8.3, and this yields the formula

$$[\mathrm{coef}\ ((h_1, l_1), \ldots, (h_m, l_m)); \sigma](R(\mu_{b_{1,1},b_{1,2},\ldots,b_{k,1},b_{k,2}})) =$$

$$= \prod_{\substack{1 \le h \le k \\ \textit{such that} \\ L_h \neq \emptyset}} \{ \prod_{\substack{B \textit{ block of } \sigma \\ \textit{such that} \\ B \subseteq L_h}} [\text{coef } (|B|/2)](f_h) \} \tag{8.11}$$

(where $f_1, \ldots, f_k$ are the series of one variable discussed in Remark 8.2).

We now start looking at the term indexed by σ in the sum (8.6). From the parallel discussion made above, we see that we have to consider three cases: (a) $\sigma \in NC(m) \setminus NC_{h-acc}(m)$; (b) $\sigma \in NC_{h-acc}(m) \setminus NC_{\varepsilon-alt}(m)$; and (c) $\sigma \in NC_{h-acc}(m) \cap NC_{\varepsilon-alt}(m)$. More precisely, we have to show that the term indexed by σ in (8.6) is zero in the cases (a) and (b), and is given by the right-hand side of (8.11) in the case (c).

Case (a) is easy. Indeed, by using the freeness of $\{p_{1,1}, p_{1,2}\}, \ldots, \{p_{k,1}, p_{k,2}\}$ and Theorem 3.6, it is easily seen that in this case [coef $((h_1, l_1), \ldots, (h_m, l_m)); \sigma]$ $(R(\mu_{p_{1,1}, p_{1,2}, \ldots, p_{k,1}, p_{k,2}}))$ equals zero.

Case (c) is also easy. Indeed, in this case the h-acceptability of σ implies that [coef $((h_1, l_1), \ldots, (h_m, l_m)); \sigma]$ $(R(\mu_{p_{1,1}, p_{1,2}, \ldots, p_{k,1}, p_{k,2}}))$ has an expression of the kind shown in the right-hand side of (8.10) (with the b's replaced by p's); and after that, due to the assumption that $\sigma \in NC_{\varepsilon-alt}(m)$, this expression can be prelucrated by using Lemma 8.3, and yields exactly the right-hand side of (8.11). On the other hand, in case (c) we also have $C_Q(\sigma) \in NC_{\varepsilon-alt}(m)$ (by Proposition 7.7); by using this fact, and by following how $(\lambda_1, \ldots, \lambda_m)$ is determined by $\varepsilon = (l_1, \ldots, l_m)$ (see Remark 8.7), one sees immediately that we have [coef $(\lambda_1, \ldots, \lambda_m); C_Q(\sigma)]$ $(M(\mu_{u,u^{-1}}))$ =1.

Finally, let us consider the case (b). The h-acceptability of σ implies that (similarly to (8.10)) we have the formula

$$[\text{coef } ((h_1, l_1), \ldots, (h_m, l_m)); \sigma](R(\mu_{p_{1,1}, p_{1,2}, \ldots, p_{k,1}, p_{k,2}})) =$$

$$= \prod_{\substack{1 \le h \le k \\ \textit{such that} \\ L_h \neq \emptyset}} \{ \prod_{\substack{B \textit{ block of } \sigma \\ \textit{such that} \\ B \subseteq L_h}} [\text{coef } ((h_1, l_1), \ldots, (h_m, l_m))|B](R(\mu_{p_{h,1}, p_{h,2}})) \}. \tag{8.12}$$

We have to show that either the product in (8.12) is zero, or [coef $(\lambda_1, \ldots, \lambda_m); C_Q(\sigma)]$ $(M(\mu_{u,u^{-1}}))$ is zero. We will obtain this from the hypothesis that $\sigma \notin NC_{\varepsilon-alt}(m)$, in the equivalent formulation that comes out by negating Proposition 8.11. That is, we know that because $\sigma \notin NC_{\varepsilon-alt}(m)$, one of the following three things must happen:

1^o either: σ has a block B with one element. In this case the product in (8.12) contains a factor which is a coefficient of length one of the series $R(\mu_{p_{h,1},p_{h,2}})$, $1 \le h \le k$; but any such coefficient is zero, because the pairs $(p_{h,1}, p_{h,2})$ are diagonally balanced.

2^o or: σ has a block B such that $|B|$ is odd and $(l_1, \ldots, l_m)|B$ is a cyclic permutation of $(1, 2, \ldots, 1, 2, 1)$ or of $(2, 1, \ldots, 2, 1, 2)$. Let $h \in \{1, \ldots, k\}$ be such that B is contained in the level set L_h. Then [coef $((h_1, l_1), \ldots, (h_m, l_m))|B](R(\mu_{p_{h,1},p_{h,2}}))$ (which is the same thing as [coef $((h, l_1), \ldots, (h, l_m))|B](R(\mu_{p_{h,1},p_{h,2}}))$, by the definition of L_h), is zero - because $(p_{h,1}, p_{h,2})$ is diagonally balanced, and by Remark 5.2.

3^o or: $C_Q(\sigma)$ has a block containing an odd number of elements. But then it is clear that [coef $(\lambda_1, \ldots, \lambda_m); C_Q(\sigma)](M(\mu_{u,u^{-1}}))$ =0. This concludes the discussion of case (b), and the proof. **QED**

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